Norwegian University of Science and Technology
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Exercise set 8
Exercise session 2017-10-19

1 (Exercise 7.2.3 p 74 in Krogstad)

(a) The most common model for the traffic of cars along a road leads to a dimensionless flux of cars of the form $j = \rho(1-\rho)$. Describe how this model is set up. State the hyperbolic equation the model leads to (where no cars enter or leave the road). When will the solution develop shocks?

Consider the model in (a). Between $x = 0$ and $x = 1$ there is a reduction in the speed limit such that the maximum speed reduces to $1/2$, while the maximum density remains the same. We assume that the same type of relation between the car velocity and the density also applies for this part.

(b) Which condition on the flux of cars has to hold in $x = 0$ and $x = 1$? Find the solution $\rho(x,t)$ for $t > 0$ and all x when

$$\rho(x,0) = \begin{cases} 1/2, & x < 1, \\ 0, & x > 1. \end{cases}$$

Hint: The density ρ is constant between 0 and 1 for all $t \geq 0$.

2 Solve the following initial/boundary value problems for $t \in [0,1]$,

(a)

$$\begin{aligned} u_t + \left(\frac{1}{m}u^m\right)_x &= 0, & x > 0, t > 0, \\ u(0,t) &= 1, & x = 0, t > 0, \\ u(x,0) &= \begin{cases} 0, & x \in [0,1], \\ 1, & x > 1. \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} u_t + f(u)_x &= 0, & x > 1, t > 0, \\ f(u) &= 0, & x = 1, t > 0, \\ u(x,0) &= 0, & x > 1, t = 0, \end{aligned}$$

where

$$f(u) = \frac{1}{3}(u^2-1)(u-2).$$

3 The goal of this exercise is to derive equations determining the flow of an ideal gas in a long, thin horizontal pipeline. That is, assume that the velocity u depends on x

and t only, and that there is no movement in y- and z-direction. For simplicity we are going to assume that the temperature T is constant, and we will ignore viscosity ($\mu = 0$). For an ideal gas the pressure p is related to the density ρ by the ideal gas law $p = \rho CT$, where C is a constant depending on the molar mass of the gas.

(a) State the conservation laws for mass and momentum on integral form of the system described above.
Hint: The only force acting on the gas is pressure.

(b) Derive the conservation laws for mass and momentum on differential form.

(c) When pressure differences increase, the temperature will no longer be constant. What are the conservation laws in this case?
Hint: The energy density E is given by $E = \frac{1}{2}\rho v^2 + \rho e$, where v is the velocity and e is the specific internal energy. For an ideal gas the specific internal energy is given by $e = \hat{c}\rho T$, where $\hat{c}$ is the specific heat capacity.

The above models are extensively applied in industry and research, by for example SINTEF.

Extra One-dimensional conservation laws often lead to first order partial differential equations of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0.$$

The Riemann problem for these equations consists of finding $\rho(x,t)$ for $t > 0$ when

$$\rho(x,0) = \begin{cases} \rho_1 \,, & x < 0, \\ \rho_2 \,, & x > 0. \end{cases}$$

(a) Summarize which elementary situations may arise, and how we can find solutions in these cases.

Consider a viscous fluid with density ρ flowing down an inclined plane with angle α due to gravitational pull. Let the x axis point downward along the inclined plane, and let the y axis point upward, orthogonally to the plane. The velocity of the fluid is assumed to be pointed in the x direction with speed $u = u(y)$ and $u(0) = 0$.The shear stress τ in the fluid on a plane parallel to the inclined plane is given by $\tau(y) = \mu\frac{du}{dy}$. Shear stress has the unit force per area. The constant μ is the fluid's viscosity.

(b) Assume the flow is stationary and that the thickness of the fluid is h everywhere. Choose a control volume between y and the surface h and show, using a simple balance of forces, that $\tau = (h - y)\rho g \sin\alpha$. Then show that $Q(h)$, the total volume passing a given point on the plane per time unit, is given by

$$Q(h) = \frac{\rho g \sin\alpha}{3\mu} h^3. \tag{1}$$

(c) Assume equation 1 holds even if h varies. State the conservation law for the fluid in differential form. Show that given scales H and X for h and x respectively, there exists a scale for t such that the conservation law states

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\frac{h^3}{3}\right) = 0. \tag{2}$$

(d) We will solve equation 2 in the first quadrant when

$$h(x,t) = \begin{cases} 0 & \text{for } x > 0,\ t = 0, \\ \sqrt{t} & \text{for } x = 0,\ t \geq 0. \end{cases}$$

Sketch the characteristics, and show that a point (x, t) lies on the characteristic from the point $(0, t_0)$ if and only if

$$t_0 = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - x}.$$

(e) The solution in (d) develops a shock, $(s(t), t)$, where the characteristic from $\left(0, \frac{t}{2} + \sqrt{\frac{t^2}{4} - s(t)}\right)$ meets a characteristic from the x axis. Find a differential equation for the propagation speed of the shock, verify that $s(t) = 5t^2/36$ is a solution, and finally state the complete solution to the problem.