Norwegian University of Science and Technology
Department of Mathematical Sciences

Exercise set 9
Exercise session 2017-10-26

1 Let $\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)e^{-ix\xi}\,\mathrm{d}x$ be the Fourier transform. Recall that

$$\mathcal{F}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)(\xi) = i\xi\mathcal{F}(u),$$
$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-z)g(z)\,\mathrm{d}z,$$
$$\mathcal{F}(f * g) = \sqrt{2\pi}\mathcal{F}(f)\mathcal{F}(g),$$
$$\mathcal{F}\left(e^{-ax^2}\right)(\xi) = \frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}.$$

Note that we require $\lim_{|\xi|\to\infty} \hat{u}(\xi) = 0$. Assume this when necessary.

(a) Solve the initial value problem for the heat equation using the Fourier transform.

(b) Find the solution of

$$u_t(x,t) = c^2 u_{xx}(x,t) + f(x,t), \qquad x \in \mathbb{R}, t > 0$$
$$u(x,0) = u_0(x), \qquad t = 0, x \in \mathbb{R}.$$

2 (Problem 7.1.6 p. 71 in Krogstad)

(a) Define the fundamental solution $c_F(x,t)$ of the equation

$$\frac{\partial c}{\partial t} = \kappa \frac{\partial^2 c}{\partial x^2}, \tag{1}$$

where $c(x,t)$ is the concentration of a substance and κ is a positive constant. The fundamental solution represents a unit discharge in the point $x = 0$ at time $t = 0$.

(b) Show that the total amount, $\int_{-\infty}^{\infty} c_F(x,t)\,\mathrm{d}x$, remains equal to 1 for all $t > 0$. It is reasonable to consider $c_F(x,t)$ as a probability density on $\mathbb{R}$. What is in this case the mean value, μ, and standard deviation, σ?

(c) Which condition on the diffusive $\mathbf{j}_d$ must hold at x_0 if there is a dense wall there, so that all diffusion occurs to the right of the wall ($x > x_0$).

(d) Show (or argue) that if x_0 in (g) is smaller than 0, it is possible to write the solution for a unit emission at $x = 0$ for $t = 0$ as

$$c(x,t) = c_f(x,t) + c_f(x - 2x_0, t), \qquad x \geq x_0, t > 0.$$

3 In a porous medium in $\mathbb{R}^1$, a gas can occupy only the pore volume, a fraction $\phi \in (0,1)$ of the total volume. The mass flux is given by Darcy's law,

$$j(x,t) = -\rho(x,t)\frac{k}{\mu}p_x(x,t),$$

where ρ is the mass density of the gas, μ the viscosity, and k the permeability. We also assume an ideal gas, i.e.

$$p = \rho RT,$$

where R is the gas constant and T is the temperature, which is assumed to be constant.

(a) Use conservation of mass to show that

$$\phi\rho_t + j_x = 0,$$

and then that

$$\rho_t = K(\rho\rho_x)_x. \tag{2}$$

What is K?

After scaling, (2) takes the form

$$\rho_t = 2(\rho\rho_x)_x = (\rho^2)_{xx}. \tag{3}$$

Let $C > 0$ and

$$\rho_F(x,t) = \begin{cases} Ct^{-\frac{1}{3}} - \frac{1}{12}x^2t^{-1} & \text{for } |x|^2 \le 12Ct^{\frac{2}{3}}, \\ 0 & \text{for } |x|^2 \ge 12Ct^{\frac{2}{3}}. \end{cases} \tag{4}$$

This functions is plotted in figure 1. Note that ρ_F is non-negative, even and continuous.

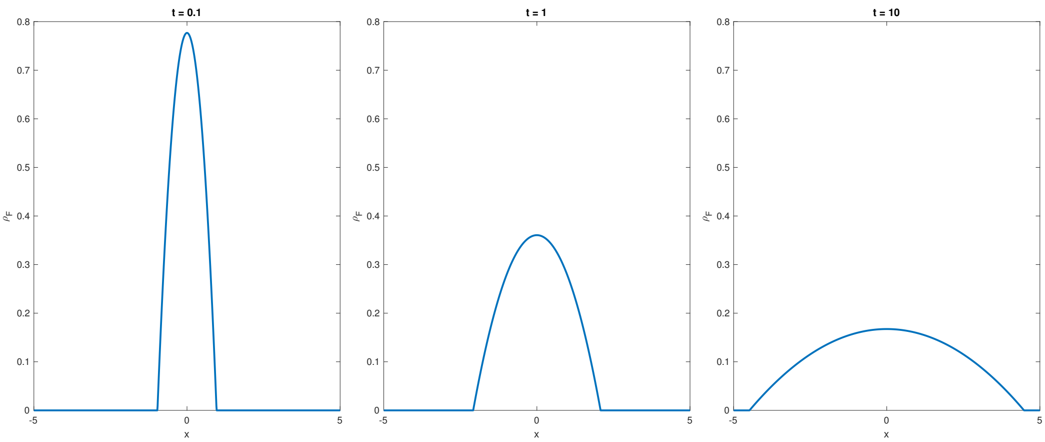


Figure 1: The function ρ_F plotted at times $t = 0.1$, $t = 1$ and $t = 10$, with $C = 3^{\frac{1}{3}}/4$.

(b) Show that

$$\int_{-\infty}^{\infty} \rho_F(x,t)\,dx = 1 \qquad \text{when } C = \frac{3^{\frac{1}{3}}}{4}, \tag{5}$$

and that ρ_F satisfies (3) for $|x|^2 \neq 12Ct^{\frac{2}{3}}$.
Hint: Consider the regions $|x|^2 < 12Ct^{\frac{2}{3}}$ and $|x|^2 > 12Ct^{\frac{2}{3}}$ separately.

(c) Let $f(x)$ be continuous. Let $C = \frac{3^{\frac{1}{3}}}{4}$ in (4) and show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} \rho_F(x,t) f(x)\,dx = f(0).$$

Explain why $\rho_F(x,t)$ is a fundamental solution of (3).
Hint: Use (5), the fact that $\rho_F \geq 0$, and that

$$\lim_{r\to 0} \max_{|x|<r} |f(x) - f(0)| = 0$$

for any continuous function f.

It can be shown that the scaled equation (3) can also be used for the scaled height $h = h(x,t)$ in a two-dimensional model of a groundwater reservoir, i.e.

$$h_t = (h^2)_{xx}.$$

(d) Assume that the ground is completely dry, and there is a well at $x = 0$, which at $t = 0$ is filled up completely. The mass of the added water is such that

$$\int_{-\infty}^{\infty} h(x,0)\,dx = 2.$$

What is the extension of wet ground at the dimensionless (scaled) time $t = 10$?
Hint: Assume that the well is a point source. Explain why you can use (4). What must C be in this case?

4 (Problem 7.2.6 p. 77 in Krogstad) Discharge of contaminants into a river will be transported with the flow (convection) and spread due to turbulence mixing and varying water velocity (diffusion). Consider a one-dimensional river with mean flow U and the diffusion coefficient κ.

(a) Derive the expression for the flux of contaminants under these simple conditions, and find a length scale of the extent of an instantaneous point discharge after this has been carried a length L down the river by means of the current.

At the point $x = 0$ there is a continuous discharge of a substance A so that the concentration in the river becomes $a(x,t)$. The substance A is converted into substance B with constant rate μ. Thus, for a water sample from the river we would have

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mu a.$$

The substance B decays with rate λ, and for the same water sample, the concentration $b(x,t)$ of B fulfills

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \mu a - \lambda b.$$

(b) State the conservation laws for A and B on the integral and differential form.

(c) The discharge at $x = 0$ takes place at a constant rate q_0 (amount per time unit). Neglect diffusion and decide how far down the river the concentration of the substance B is at its highest when we assume that $\lambda = \mu$.
Hint: The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} + ky = \mathrm{e}^{-kt}$ has the general solution $y(t) = C_1\mathrm{e}^{-kt} + t\mathrm{e}^{-kt}$.

5 A river flows into an ocean basin. The river brings sand and clay so that the basin is filled up over time. We shall formulate a simple one-dimensional model for how the basin is filled, and we assume that the basin spans from $x = 0$ to $x = +\infty$ and has a constant depth h at $t = 0$. Conditions across (in the y-direction) are constant.

The amount of sand and clay which settle on the bottom per time and area unit is $Q(x,t)$. We write the depth $z = b(x,t)$, $x \geq 0$, $t \geq 0$, and let $b(x,t) \leq 0$. If the bottom tilts, the particles on the bottom will continue to move, and it has been found that the mass flux will be proportional to the slope, that is, the volume flux may be written

$$j = -k\frac{\partial b}{\partial x}. \tag{6}$$

(a) Write the conservation equation in integral form for a part of the bottom, $x_0 \leq x \leq x_1$, and show that we for the differential formulation obtain an equation for $b(x,t)$ which is identical to the heat equation,

$$b_t = kb_{xx} + q. \tag{7}$$

A similarity solution b of (7) is a solution of the form

$$b(x,t) = t^{\alpha}\phi(\frac{x}{t^{\beta}}) \qquad \text{for some } \alpha, \beta \in \mathbb{R}.$$

(b) Assume that the river deposits all sand and clay at $x = 0$ (i.e. $q = 0$ for $x > 0$), and that the effect of this deposition is to maintain the depth 0 at $x = 0$:

$$b(0,t) = 0 \qquad \text{for all } t > 0.$$

Show that, in this case, there is a similarity solution of (7). Determine this solution.
(Hint: The equation

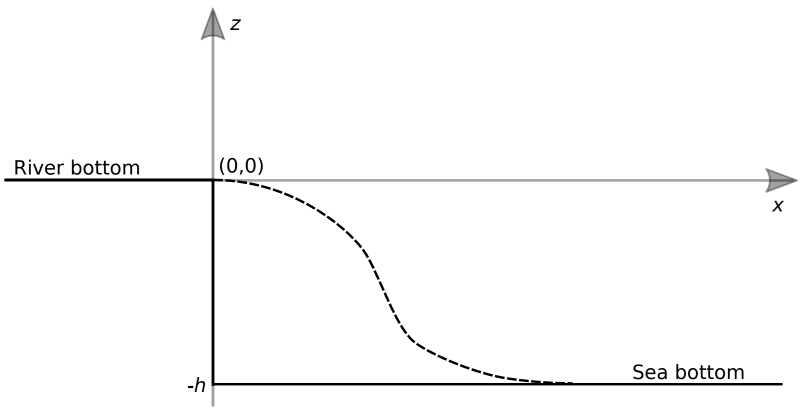
$$\frac{d^2y}{d\eta^2} + \frac{\eta}{2}\frac{dy}{d\eta} = 0$$

has the general solution $C_1 + C_2 \operatorname{erf}(\eta/2)$, where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-s^2\right) ds$.)

A more realistic scenario is that as time goes by (geologic timescale), the shore $s(t)$ will move forward with time (see the figure below). Suppose we have a constant volume flow q_0 in the basin per unit time and unit width at the shore, and that all sand and clay is added in this way. The solution for $b(x,t)$ will be stationary with respect to the shore, ie. $b(x,t)$ may be written by means of a function b_0 so that

$$b(x,t) = \begin{cases} 0 & x \leq s(t) = Ut + x_0 \\ b_0(x - Ut - x_0) & x > Ut + x_0 \end{cases}$$

(c) Determine the velocity U and the solution in this case.


z
River bottom
(0,0)
x
-h
Sea bottom