Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195
Mathematical Modelling
Autumn 2017

Solutions to exercise set 9

1 (a) The initial value problem for the heat equation is $u_t = c^2 u_{xx}, u(x,0) = u_0(x), c \neq 0$. We assume that $u, u_{xx} \in L^2(\mathbb{R})$ for each t. The Fourier transform in x gives that the IVP is

$$\begin{aligned}
\hat{u}_t(\xi,t) &= -c^2\xi^2\hat{u}(\xi,t), && \text{solving yields}\\
\hat{u}(\xi,t) &= \hat{u_0}(\xi)e^{-c^2\xi^2 t}, && \text{we take the inverse Fourier transform}\\
u(x,t) &= u_F(\cdot,t) * u_0(\cdot)(x,t) = \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}\left(e^{-c^2\xi^2 t}\right) * u_0(x)\\
&= \frac{1}{\sqrt{4\pi c^2 t}}\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4tc^2}} u_0(z)\,\mathrm{d}z.
\end{aligned}$$

(b) Fourier transform gives the ODE $\hat{u}_t = -c^2\xi^2\hat{u} + \hat{f}$. Multiply the equation by the integrating factor $e^{c^2\xi^2 t}$ and use the product rule of differentiation to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{c^2\xi^2 t}\hat{u}\right) = e^{c^2\xi^2 t}\hat{f}.$$

We integrate

$$\hat{u}(\xi,t) = e^{-c^2\xi^2 t}\hat{u_0}(\xi) + \int_0^t e^{-c^2\xi^2(t-s)}\hat{f}(\xi,s)\,\mathrm{d}s.$$

Inverse Fourier transform gives

$$u(x,t) = u_F(\cdot,t) * u_0(\cdot)(x,t) + \int_0^t u_F(\cdot,t-s) * f(\cdot,s)\,\mathrm{d}s.$$

2 (a) We solve $c_t = \kappa c_{xx}$ (for example via the Fourier transform), and find that

$$c(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4\kappa t}} c_0(z)\,\mathrm{d}z,$$

for some initial function c_0. Our $c_0(x) = \delta_0(x)$, and thus $c_F = \frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{x^2}{4\kappa t}}$.

(b) We have that $\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$. Taking $u = \frac{x}{\sqrt{4\kappa t}}$ and substituting gives

$$\int_{-\infty}^{\infty} c_F(x,t)\,\mathrm{d}x = \frac{1}{\sqrt{4\pi\kappa t}}\sqrt{4\kappa t}\int_{-\infty}^{\infty} e^{-u^2}\,\mathrm{d}u = 1.$$

The mean value, μ, is given by $\mu(t) = \int x c_F(x,t)\,\mathrm{d}x$. This integral is defined for all $t > 0$. Moreover c_F is an even function of x, while x is odd. This gives

$\mu = 0$ for all $t > 0$. The standard deviation, σ, when the mean value is zero is given by $\sigma(t)^2 = \int x^2 c_F(x,t)\,\mathrm{d}x$. We compute the integral

$$\begin{aligned}\sigma(t)^2 &= \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4\kappa t}} x^2\,\mathrm{d}x \\ &= \frac{(4\kappa t)^{\frac{3}{2}}}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} u^2 e^{-u^2}\,\mathrm{d}u \\ &= 2\kappa t.\end{aligned}$$

(c) We have $\mathbf{j}_d(x_0) = c_x(x_0) = 0$.

(d) Linearity and translation invariance ensures that $c(x,t) = c_F(x,t) + c_F(x - 2x_0, t)$ solves the heat equation. Furthermore, as c_F is even, $(c_F)_x$ has to be odd and thus $c_x(x_0,t) = (c_F)_x(x_0,t) + (c_F)_x(-x_0,t) = 0$. Moreover the integral $\int_{x_0}^{\infty} c(x,t)\,\mathrm{d}x = \int_{x_0}^{\infty}\big(c_F(x,t) + c_F(x-2x_0,t)\big)\,\mathrm{d}x = \int_{-\infty}^{\infty} c_F(x,t)\,\mathrm{d}x$. As $t \to 0$ we have that $c \to \delta_0$ in some sense.

3 (a) We look at an interval $[x, x+\Delta x]$ on $\mathbb{R}$, and get that the conservation of mass in the interval is

$$\frac{\text{change of mass in the interval}}{\text{time}} = \text{flux in at } x - \text{flux out at } (x+\Delta x),$$
$$\Downarrow$$
$$\frac{d}{dt}\int_x^{x+\Delta x} \phi\rho(y,t)\,dy = j(x,t) - j(x+\Delta x, t).$$

Applying Leibniz's rule to the left hand side, we get

$$\int_x^{x+\Delta x} \phi\rho_t(y,t)\,dy = -\int_x^{x+\Delta x} j_x(y,t)\,dy.$$

In general, we have for a continuous function f that

$$\lim_{\Delta x\to 0}\left(\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(x)\,dx\right) = f(x_0),$$

and hence

$$\begin{aligned}\phi\rho_t(x,t) &= \lim_{\Delta x\to 0}\left(\frac{1}{\Delta x}\int_x^{x+\Delta x}\phi\rho_t(y,t)\,dy\right) \\ &= -\lim_{\Delta x\to 0}\left(\frac{1}{\Delta x}\int_x^{x+\Delta x} j_x(y,t)\,dy\right) = -j_x(x,t).\end{aligned}$$

Furthermore,

$$\begin{aligned}j_x(x,t) &= \frac{\partial}{\partial x}\left(-\rho\frac{k}{\mu}p_x\right) = \frac{\partial}{\partial x}\left(-\rho\frac{k}{\mu}\frac{\partial}{\partial x}(\rho RT)\right) \\ &= -\frac{kRT}{\mu}(\rho\rho_x)_x,\end{aligned}$$

and thus we have

$$\rho_t = K(\rho\rho_x)_x$$

with

$$K = \frac{kRT}{\mu\phi}.$$

(b) Since ρ_F is even and $\rho_F = 0$ for $x^2 > 12Ct^{\frac{2}{3}}$, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} \rho_F(x,t)\,dx &= 2\int_0^{\sqrt{12Ct^{2/3}}} \rho_F(x,t)\,dx \\
&= 2\int_0^{\sqrt{12Ct^{2/3}}} (Ct^{-\frac{1}{3}} - \frac{1}{12}x^2t^{-1})\,dx \\
&= 2\left[Ct^{-\frac{1}{3}}x - \frac{1}{12}\frac{1}{3}t^{-1}x^3\right]_0^{\sqrt{12Ct^{2/3}}} \\
&= 2\left(Ct^{-\frac{1}{3}}\sqrt{12Ct^{\frac{2}{3}}} - \frac{1}{12}\frac{1}{3}t^{-1}(12Ct^{\frac{2}{3}})^{\frac{3}{2}}\right) \\
&= 2\left(\sqrt{12}C^{\frac{3}{2}} - \frac{1}{3}\sqrt{12}C^{\frac{3}{2}}\right) = \frac{4}{3}\sqrt{12}C^{\frac{3}{2}}.
\end{aligned}$$

With $C = \frac{3^{\frac{1}{3}}}{4}$, we get

$$\int_{-\infty}^{\infty} \rho_F(x,t)\,dx = \frac{4}{3}\sqrt{12}\left(\frac{3^{\frac{1}{3}}}{4}\right)^{\frac{3}{2}} = \frac{8}{\sqrt{3}}\frac{\sqrt{3}}{8} = 1.$$

Following the hint and considering the regions separately, we immediately see that ρ_F satisfies the given equation in the regions $|x|^2 > 12Ct^{\frac{2}{3}}$. In the region $|x|^2 < 12Ct^{\frac{2}{3}}$, we have

$$\begin{aligned}
(\rho_F)_t &= -\frac{1}{3}Ct^{-\frac{4}{3}} + \frac{1}{12}x^2t^{-2}, \\
(\rho_F)_x &= -\frac{1}{6}xt^{-1}, \\
(\rho_F)_{xx} &= -\frac{1}{6}t^{-1},
\end{aligned}$$

and hence

$$\begin{aligned}
2(\rho_F(\rho_F)_x)_x &= 2(\rho_F)_x^2 + 2\rho_F(\rho_F)_{xx} \\
&= 2\frac{1}{6^2}x^2t^{-2} + 2\left(-\frac{1}{6}Ct^{-\frac{1}{3}-1} + \frac{1}{6}\frac{1}{12}x^2t^{-2}\right) \\
&= \frac{1}{12}x^2t^{-2} - \frac{1}{3}Ct^{-\frac{4}{3}} \\
&= (\rho_F)_t,
\end{aligned}$$

and thus the equation is satisfied by ρ_F also in this region.

(c) Because $\int_{-\infty}^{\infty} \rho_F\,dx = 1$, we have that

$$f(0) = \int_{-\infty}^{\infty} \rho_F(x,t)f(0)\,dx$$

and applying this and that $\rho_F \geq 0$, we get

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \rho_F(x,t)f(x)\,dx - f(0)\right| &= \left|\int_{-\infty}^{\infty} \rho_F(x,t)(f(x)-f(0))\,dx\right| \\ &\leq \int_{-\infty}^{\infty} \rho_F(x,t)\,|f(x)-f(0)|\,dx \\ &= \int_{-\sqrt{12Ct^{2/3}}}^{\sqrt{12Ct^{2/3}}} \rho_F(x,t)\,|f(x)-f(0)|\,dx \\ &\leq \max_{|x|\leq\sqrt{12Ct^{2/3}}} |f(x)-f(0)| \cdot \int_{-\infty}^{\infty} \rho_F(x,t)\,dx \\ &= \max_{|x|\leq\sqrt{12Ct^{2/3}}} |f(x)-f(0)|.\end{aligned}$$

Now we have

$$\lim_{t\to 0}\left|\int_{-\infty}^{\infty} \rho_F(x,t)f(x)\,dx - f(0)\right| \leq \lim_{\sqrt{12Ct^{2/3}}\to 0}\left(\max_{|x|\leq\sqrt{12Ct^{2/3}}} |f(x)-f(0)|\right) = 0,$$

since f is continuous, and $t \to 0$ implies $\sqrt{12Ct^{2/3}} \to 0$.
A fundamental solution is a solution with initial data $\rho_{F,0} = \delta_0$, where the delta function $\delta_0(x)$ is a function such that

$$\int_{-\infty}^{\infty} f(x)\delta_0(x)\,dx = f(0)$$

for any continuous function f. We have that

$$\int_{-\infty}^{\infty} \rho_F(x,0)f(x)\,dx = \lim_{t\to 0}\int_{-\infty}^{\infty} \rho_F(x,t)f(x)\,dx = f(0),$$

and hence the initial solution $\rho_F(x,0)$ of ρ_F is by definition a delta function.

(d) The equation is given, and as initial solution we follow the hint and use a positive point source with integral 2. Thus we get

$$\begin{cases} h_t &= (h^2)_{xx}, \\ h(x,0) &= 2\delta_0. \end{cases} \tag{1}$$

Following a similar deduction as when solving problem (b), we get that

$$\int_{-\infty}^{\infty} \rho_F(x,t)\,dx = 2 \qquad \text{when } C = \left(\frac{3}{16}\right)^{\frac{1}{3}}.$$

We therefore have that equation (4) in the problem set is a solution to (1) above, with $C = \left(\frac{3}{16}\right)^{\frac{1}{3}}$. We then have that $h > 0$ for $|x|^2 < 12Ct^{\frac{2}{3}}$, and hence the extension of wet ground at $t = 10$ is given by

$$|x| = \sqrt{12Ct^{\frac{2}{3}}} = \sqrt{12\left(\frac{3}{16}\right)^{\frac{1}{3}} 10^{\frac{2}{3}}} = 12^{\frac{1}{2}}\left(\frac{3}{16}\right)^{\frac{1}{6}} 10^{\frac{1}{3}} \approx 5.65.$$

4 (a) Denote by c the concentration of contaminants. There are two sources of flux. The first is the diffusive flux, given by Fick's law:

$$j_d = -\kappa \frac{\partial c}{\partial x}.$$

The second is the advective flux:

$$j_a = Uc.$$

Together, they form the total flux:

$$j = j_d + j_a = -\kappa \frac{\partial c}{\partial x} + Uc.$$

A point discharge will be carried a length L down the river after a time $T = L/U$. The discharge will spread out due to diffusion; the extent of this can be measured as in exercise 2c), i.e. the spread is proportional to $\sqrt{\kappa T} = \sqrt{\kappa L/U}$.

(b) Using the flux from a) and noting that the conversion of substance A into B constitutes production terms, we state the conservation laws in integral form for an interval $[x_1, x_2]$ (we omit the t-dependence for readability):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} a(x)\mathrm{d}x = \kappa\left(\frac{\partial a}{\partial x}(x_2) - \frac{\partial a}{\partial x}(x_1)\right) - U(a(x_2) - a(x_1)) - \int_{x_1}^{x_2} \mu a(x)\mathrm{d}x,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} b(x)\mathrm{d}x = \kappa\left(\frac{\partial b}{\partial x}(x_2) - \frac{\partial b}{\partial x}(x_1)\right) - U(b(x_2) - b(x_1)) + \int_{x_1}^{x_2} \mu a(x) - \lambda b(x)\mathrm{d}x.$$

To obtain the differential form of the conservation equation, we take $x_2 = x_1 + \Delta x$, divide by Δx and let $\Delta x \to 0$. This yields

$$\frac{\partial a}{\partial t} = \kappa \frac{\partial^2 a}{\partial x^2} - U\frac{\partial a}{\partial x} - \mu a,$$
$$\frac{\partial b}{\partial t} = \kappa \frac{\partial^2 b}{\partial x^2} - U\frac{\partial b}{\partial x} + \mu a - \lambda b.$$

(c) We now neglect diffusion and consider the convection-reaction equations:

$$\frac{\partial a}{\partial t} + U\frac{\partial a}{\partial x} = -\mu a, \qquad x > 0, t > 0,$$
$$\frac{\partial b}{\partial t} + U\frac{\partial b}{\partial x} = \mu a - \mu b \qquad x > 0, t > 0.$$

We need some boundary conditions. Since there is a constant rate of discharge of substance A at $x = 0$, we have a constant flux $j_A = Ua = q_0$ at $x = 0, t > 0$. Additionally, we may assume that the river is uncontaminated at $t = 0$, and that there is no discharge of substance B at $x = 0$. This gives the boundary conditions:

$$a(0,t) = \frac{q_0}{U}, \qquad t > 0$$
$$a(x,0) = 0, \qquad x > 0$$
$$b(0,t) = 0, \qquad t > 0$$
$$b(x,0) = 0, \qquad x > 0.$$

We solve the PDEs using the method of lines. Taking $z(t) = a(x(t), t)$, we get

$$\begin{aligned} \dot{z} &= -\mu z \\ \dot{x} &= U. \end{aligned}$$

Solving these ODEs yields

$$\begin{aligned} z(t) &= z(t_0)\mathrm{e}^{\mu(t-t_0)} \\ x(t) &= U(t - t_0) + x_0. \end{aligned}$$

If $t_0 = 0$, we have $z(t_0) = 0$. Otherwise, $z(t_0) = \frac{q_0}{U}$. Now, to solve for $b(x, t)$, we take $w(t) = b(x(t), t)$, observing that the characteristics for a and b are identical. This gives us the ODE for w:

$$\begin{aligned} & \dot{w} + \mu w = \mu z \\ \Rightarrow \quad & \dot{w} + \mu w = \frac{\mu q_0}{U}\mathrm{e}^{\mu(t-t_0)}, \end{aligned}$$

where we have disregarded the trivial case of characteristics starting at $t = 0$, which result in $b \equiv 0$. Now, using the hint, we obtain

$$w(t) = C_1\mathrm{e}^{\mu t} + \frac{\mu q_0}{U} t\mathrm{e}^{\mu(t-t_0)},$$

and applying the initial condition $w(t_0) = 0$, we get

$$w(t) = (t - t_0)\frac{\mu q_0}{U}\mathrm{e}^{\mu(t-t_0)},$$

yielding

$$b(U(t - t_0), t) = (t - t_0)\frac{\mu q_0}{U}\mathrm{e}^{\mu(t-t_0)}.$$

We now "invert" by reintroducing $x = U(t - t_0)$ and get

$$b(x, t) = \begin{cases} x\frac{\mu q_0}{U^2}\mathrm{e}^{\frac{\mu x}{U}}, & x < Ut \\ 0, & x > Ut. \end{cases}$$

To obtain the point of highest concentration of B, we fix a t and observe that, disregarding the case with $b = 0$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}b(x, t) &= \frac{\mu q_0}{U^2}\mathrm{e}^{\frac{\mu x}{U}}(1 - \frac{\mu}{U}x) = 0 \\ \Rightarrow x &= \frac{U}{\mu}. \end{aligned}$$

This x is attainable if $t > \frac{1}{\mu}$. Otherwise, since $\frac{\mathrm{d}}{\mathrm{d}x}b(x, t) > 0$ for $x < \frac{U}{\mu}$, the maximum is attained at $x = Ut$.

5 (a) We do the computations with a segment with width B and introduce density, flux and sources. Here we assume that the density of sand ρ is a constant. The flux then becomes ρj and the source function will be $\rho q(x; t)$. (However, both B and ρ drop out from the relations at the end, such that we could as well compute per unit width, and with $\rho = 1$).

Our control volume has width B and extends from $x = x_0$ to $x = x_1$. Thus, we obtain the general (one-dimensional) conservation law

$$\frac{d}{dt}\int_{x_0}^{x_1} \rho B\left(b\left(x,t\right)-h\right)dx+\left(-k\frac{\partial b}{\partial x}\left(x_1,t\right)+k\frac{\partial b}{\partial x}\left(x_0,t\right)\right)\rho B=\int_{x_0}^{x_1} q\left(x,t\right)\left(\rho B\right)dx.$$

or

$$\frac{d}{dt}\int_{x_0}^{x_1} \left(b\left(x,t\right)-h\right)dx+\left(-k\frac{\partial b}{\partial x}\left(x_1,t\right)+k\frac{\partial b}{\partial x}\left(x_0,t\right)\right)=\int_{x_0}^{x_1} q\left(x,t\right)dx,$$

If we let $x_1 \to x_0$ and divide by $(x_1 - x_0)$, we obtain

$$\frac{\partial}{\partial t}\left(b-h\right)=\frac{\partial b}{\partial t}=k\frac{\partial^2 b}{\partial x^2}+q.$$

(b) In this case, the source is localized at $x = 0$, such that the equation for $x > 0$ becomes just $b_t = kb_{xx}$. We scale b with h and the solution is

$$b = hf\left(x,t,k\right)$$

It is not obvious that we have a similarity solution since the depth h could be a length scale, but this length is not associated with the horizontal length. The problem is completely equivalent to a heat conduction problem where the temperature is constant and equal to T_0 at $x = 0$, and T_∞ when $x = \infty$. The temperature could then be written as $T\left(x,t\right) = T_0 + \left(T_\infty - T_0\right)\tau\left(x,t,k\right)$, and we obtain a similarity solution. Similarly to the temperature,we should be able to write the solution for b as

$$b=-h\beta\left(\frac{x}{\sqrt{kt}}\right)=-h\beta\left(\eta\right),\ \eta=\frac{x}{\sqrt{kt}},$$

where $\beta\left(0\right) = 0$ and $\beta\left(\eta\right) \to 1$ when $\eta \to \infty$. Entering this into the equation after dividing by $-h$ lead to

$$\beta_t-k\beta_{xx}=-\frac{1}{2}\frac{x}{\sqrt{k}}\frac{1}{t^{3/2}}\beta'-k\frac{1}{kt}\beta''=0,$$

or

$$\beta''+\frac{\eta}{2}\beta'=0.$$

This is the equation given in the problem. By the hint and conditions $b(0,t) = 0$, $b(\infty,t) = -h$ we see that

$$b\left(x,t\right)=-h\,\mathrm{erf}\left(\frac{x}{\sqrt{kt}}\right).$$

(c) The sand and clay volume at sea ($x \geq x_0$ here) is given by

$$\begin{aligned}V(t) &= \int_{x_0}^{\infty} B(h-b(x,t))\,dx\\ &= \int_{x_0}^{Ut-x_0} B(h-b(x,t))\,dx+\int_{Ut-x_0}^{\infty} B(h-b(x,t))\,dx\\ &= BhUt+\int_0^{\infty} b_0(y)\,dy,\end{aligned}$$

where B was determined in a). By definition, $q_0 = V'(t)$, and hence

$$U = \frac{q_0}{Bh}.$$

If we let $\eta = x - Ut - x_0$ and put b into the equation where $x > Ut + x_0$, we obtain

$$-Ub_0' = kb_0''.$$

thus

$$b_0'' + \frac{U}{k}b_0' = 0,$$

with general solution

$$b_0(\eta) = C_1 + C_2 \exp\left(-\frac{U}{k}\eta\right).$$

It is required that

$$b_0(0) = 0,$$
$$b_0(\infty) = -h,$$

such that the solution becomes

$$b_0(\eta) = h\left(\exp\left(-\frac{U}{k}\eta\right) - 1\right).$$

Introducing the original variables leads to

$$b(x,t) = \begin{cases} 0, & x \leq s(t) = Ut + x_0, \\ h\left(\exp\left(-\frac{U}{k}(x - Ut - x_0)\right) - 1\right), & x > Ut + x_0. \end{cases}$$