Norwegian University of Science and Technology
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Exam in TMA4195 Mathematical Modeling 16.12.2017 Solutions

Problem 1

a) Here x, y are two populations varying with time t.
The equation for $\dot{x}$ consists of:

$rx(1-\frac{x}{K})$ logistic growth term of rate r and capacity K,

$-\dfrac{axy}{c+x}$ death term of rate a and dependent on both x and y,

while the equation for $\dot{y}$ consists of:

$-my$ death term, of rate a,

$+\dfrac{bxy}{c+x}$ growth term of rate b and dependent on both x and y.

Possible models include:

- x prey, y predators,
- x humans, y bacteria,
- x lemmings, y foxes,
- x fish, y fishermen.

b) We have

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = F(x,y) := \begin{bmatrix}x(2-x-\frac{y}{1+x})\\ y(-1+\frac{2x}{1+x})\end{bmatrix} \tag{1}$$

Equilibrium points (x_e, y_e) are constant solutions of (1) and therefore the solutions of

$$x\Big(2-x-\frac{y}{1+x}\Big)=0,$$
$$y\Big(-1+\frac{2x}{1+x}\Big)=0.$$

The second equation is satisfied exactly when $y = 0$ or $x = 1$. For $y = 0$, the first equation is satisfied exactly when $x = 0$ or $x = 2$, while if $x = 1$ the first equation is satisfied only for $y = 2$. Thus the equilibrium points are given by:

$$(0,0), (2,0), (1,2).$$

OBS: By carelessly multiplying the equations with $(1 + x)$, one might conclude that $(-1, 0)$ is also an equilibrium point, which is not the case (check!).

To calculate the stability of the points we look at the Jacobian of F:

$$DF(x,y) := \begin{bmatrix} 2(1-x) - \frac{y}{(1+x)^2} & -\frac{x}{1+x} \\ \frac{2y}{(1+x)^2} & -1 + \frac{2x}{1+x} \end{bmatrix}$$

We calculate the eigenvalues λ_1, λ_2 of DF in the different equilibrium points:

$$\begin{aligned} DF(0,0) &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} && \text{where } \lambda_1 = 2,\ \lambda_2 = -1, \\ DF(2,0) &= \begin{bmatrix} -2 & 2/3 \\ 0 & 1/3 \end{bmatrix} && \text{where } \lambda_1 = -2,\ \lambda_2 = 1/3, \\ DF(1,2) &= \begin{bmatrix} -1/2 & -1/2 \\ 1 & 0 \end{bmatrix} && \text{where } \lambda_1 = -\frac{1}{4} + i\frac{\sqrt{7}}{4}, \quad \lambda_2 = -\frac{1}{4} - i\frac{\sqrt{7}}{4}. \end{aligned}$$

The eigenvalues of the last matrix can be find by solving

$$\det(DF(1,2) - \lambda I) = 0 \quad \Leftrightarrow \quad (-1/2 - \lambda)(-\lambda) - (-1/2) = 0 \quad \Leftrightarrow \quad 2\lambda^2 + \lambda + 1 = 0.$$

An equilibrium point is stable if $\max\{\mathrm{Re}\lambda_1, \mathrm{Re}\lambda_2\} < 0$ and unstable if $\max\{\mathrm{Re}\lambda_1, \mathrm{Re}\lambda_2\} > 0$. Consequently we see that $(0,0)$ and $(2,0)$ are unstable while $(1,2)$ is stable.

Problem 2 Performing regular perturbation we write y as

$$y(x) = y_0(x) + \varepsilon y_1(x) + \mathcal{O}(\varepsilon^2),$$

where y_0, y_1 are not dependent on ε. As f is smooth we can Taylor expand f about 0 to obtain

$$f(\varepsilon y) = f(0) + f'(0)\varepsilon y + \mathcal{O}(\varepsilon^2),$$

which in turn can be written as

$$\begin{aligned} f(\varepsilon y) &= f(0) + f'(0)\varepsilon\Big(y_0(t) + \varepsilon y_1(t) + \mathcal{O}(\varepsilon^2)\Big) + \mathcal{O}(\varepsilon^2) \\ &= f(0) + f'(0)\varepsilon y_0(t) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Inserting these two representations in the differential equation we obtain

$$y_0'' + \varepsilon y_1'' + f(0) + f'(0)\varepsilon y_0 + \mathcal{O}(\varepsilon^2) = 0,$$

while the initial conditions become

$$\begin{aligned} y_0(0) + \varepsilon y_1(0) + \mathcal{O}(\varepsilon^2) &= 0, \\ y_0(1) + \varepsilon y_1(1) + \mathcal{O}(\varepsilon^2) &= \varepsilon. \end{aligned}$$

Equating terms of different order in ε, we obtain the system of equations:

$$\begin{aligned} &\mathcal{O}(1): && y_0'' + f(0) = 0, && y_0(0) = 0, \quad y_0(1) = 0, \\ &\mathcal{O}(\varepsilon): && y_1'' + f'(0)y_0 = 0, && y_0(0) = 0, \quad y_1(1) = 1. \end{aligned}$$

Solving first for y_0 we obtain

$$y_0(x) = -\frac{1}{2}f(0)x^2 + Ax + B,$$

where we must take $A = \frac{1}{2}f(0)$ and $B = 0$ to satisfy the boundary conditions for y_0. We can now insert for y_0 in the expression for y_1 to obtain

$$\begin{aligned} y_1'' + \frac{1}{2}f'(0)f(0)(x - x^2) &= 0, \\ \Longrightarrow y_1(x) &= \frac{1}{2}f'(0)f(0)\Big(\frac{x^4}{12} - \frac{x^3}{6}\Big) + Cx + D, \end{aligned}$$

and the boundary conditions gives $C = 1 + \frac{1}{24}f'(0)f(0)$ and $D = 0$. We conclude that

$$\begin{aligned} y(x) &= y_0(x) + \varepsilon y_1(x) + \mathcal{O}(\varepsilon^2) \\ &= \frac{1}{2}f(0)(x - x^2) + \varepsilon\Big(x + \frac{1}{24}f'(0)f(0)(x^4 - 2x^3 + x)\Big) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Problem 3

We set $z(t) = \rho(x(t), t)$ and find that the characteristic equations in this case are given by $\dot{x} = j'(z) = e^z$, $x(0) = x_0$ and $\dot{z} = 0$, $z(0) = \rho_0(x_0)$. Solving we get

$$z(t) = \rho_0(x_0) \qquad \text{and} \qquad x(t) = e^{\rho_0(x_0)}t + x_0.$$

a) In this first case, we obtain the two family of (x-)characteristics

$$\begin{cases} x(t) = x_0 + et, & x_0 < 0, \\ x(t) = x_0 + t, & x_0 > 0. \end{cases}$$

The left characteristics overtake the right characteristics (collisions) in the region $t < x < et$, so the physical solution ρ is a shock solution

$$\rho(x,t) = \begin{cases} 1, & x < s(t), \\ 0, & x > s(t), \end{cases}$$

where the shock curve $s(t)$ starts at $s = 0$ and satisfies the Rankine-Hugoniot condition:

$$\dot{s} = \frac{j(1) - j(0)}{1 - 0} = e - 1, \quad s(0) = 0 \qquad \Longrightarrow \qquad s(t) = (e-1)t.$$

b) In the second case, the two families of characteristics are given by

$$\begin{cases} x(t) = x_0 + t, & x_0 < 0, \\ x(t) = x_0 + et, & x_0 > 0. \end{cases}$$

The left characteristics are slower than the right characteristics so there is a region $t < x < et$ not reached by any characteristic (a dead sector). Hence the physical solution is given by a rarefaction wave $\rho(x,t) = \varphi(x/t)$. From the PDE for ρ we find that

$$-\frac{x}{t^2}\varphi' + e^{\varphi}\frac{1}{t}\varphi' = 0 \qquad \underset{\phi' \neq 0}{\Longrightarrow} \qquad e^{\varphi} = \frac{x}{t} \qquad \Longrightarrow \qquad \varphi(x/t) = \ln(x/t).$$

We then get the solution:

$$\rho(x,t) = \begin{cases} 0, & x < t, \\ \ln(x/t), & t < x < et, \\ 1, & et < x. \end{cases}$$

Problem 4

a) For simplicity we drop the $*$-notation in this problem and let x, y denote points in $\mathbb{R}^3$. We fix a point x in the rock formation and consider the ball $B_r := B(x,r)$, centered at x with radius $r > 0$, whose volume will be denoted $|B_r|$. By conservation of mass we have

$$\frac{d}{dt}\int_{B_r} \phi\rho\, dy = -\int_{\partial B_r} (j \cdot \hat{n}) d\sigma. \tag{2}$$

Since we have a bounded smooth domain and smooth integrands, we may change the order of differentiation and integration and use the divergence theorem to get

$$\frac{d}{dt}\int_{B_r} \phi\rho\, dy = \int_{B_r} \phi\rho_t\, dy,$$
$$\int_{\partial B_r} (j \cdot \hat{n}) d\sigma = \int_{B_r} (\nabla \cdot j)\, dy.$$

Consequently, (2) can be rewritten as

$$\int_{B_r} (\phi\rho_t + \nabla \cdot j)\, dy = 0. \tag{3}$$

Let $f := \phi\rho_t + \nabla \cdot j$, and divide by $|B_r|$, and add and subtract $f(x)$ in (3) to get

$$f(x) + \frac{1}{|B_r|}\int_{B_r} (f(y) - f(x))\, dy = 0. \tag{4}$$

Since f is continuous $f(y) \approx f(x)$ for $y \in B_r$ and $r \ll 1$, and hence for $r \ll 1$,

$$\frac{1}{|B_r|}\int_{B_r} (f(y) - f(x))\, dy \approx 0 \cdot \frac{\int_{B_r} dy}{|B_r|} \qquad \Longrightarrow \qquad \phi\rho_t(x,t) + \nabla \cdot j(x,t) = f(x) \approx 0.$$

Sending $r \to 0$ we get equality and the first equation for ρ in the problem.
[A rigorous argument which is not required for this exam, it given below:

$$\begin{aligned} \frac{1}{|B_r|}\Big|\int_{B_r} f(y) - f(x)\, dy\Big| &= \frac{1}{|B_r|}\Big|\int_{B_r}\int_0^1 \Big(\nabla f(x + s(y-x))\Big) \cdot (y-x)\, ds\, dy\Big|, \\ &\le \frac{1}{|B_r|}\int_{B_r}\int_0^1 \max_{\mathbb{R}^3} |\nabla f| r\, ds\, dy = \frac{1}{|B_r|}|B_r| \max_{\mathbb{R}^3} |\nabla f| r \to 0 \qquad \text{as } r \to 0. \quad] \end{aligned}$$

Combining Darcy's law and ideal gas law, we get

$$j = -\rho\frac{k}{\mu}\nabla p = -\rho\frac{k}{\mu}RT\nabla\rho,$$

where we used that R, T are constants. Inserting this into the equation for ρ, we obtain

$$\rho_t = \frac{kRT}{\mu\phi}\nabla(\rho\nabla\rho) =: \kappa\nabla(\rho\nabla\rho). \tag{5}$$

b) Let $x^* = Xx$, $y^* = Yy$, $z^* = Zz$, $t^* = \bar{t}t$, $\rho^* = \bar{\rho}\rho$, and

$$x, y, z, t, \rho, \rho_x, \rho_{xx}, \rho_y, \rho_{yy}, \rho_z, \rho_{zz} \sim 1 \quad \text{(scaling assumption)}.$$

By ideal gas law and $\max p^* = \bar{p}$,

$$\max \rho^* = \max \frac{p^*}{RT} = \frac{\bar{p}}{RT}.$$

Thus $\bar{\rho} = \frac{\bar{p}}{RT}$ is a natural scale for ρ^*. Good scales for x^*, y^*, z^* are

$$X = L, \quad Y = Z = 500\,\text{m} = \varepsilon L.$$

The time scale is determined introducing the scaled quatities into (5) and balancing,

$$\begin{aligned}\frac{\overline{\rho}}{\overline{t}}\rho_t &= \kappa\Big(\frac{\overline{\rho}^2}{L^2}(\rho\rho_x)_x + \frac{\overline{\rho}^2}{\varepsilon^2L^2}(\rho\rho_y)_y + \frac{\overline{\rho}^2}{\varepsilon^2L^2}(\rho\rho_z)_z\Big)\\ \implies \rho_t &= \frac{\kappa\overline{\rho}\overline{t}}{L^2}\Big((\rho\rho_x)_x + \frac{1}{\varepsilon^2}[(\rho\rho_y)_y + (\rho\rho_z)_z]\Big).\end{aligned}$$

To arrive at the desired equation, we set $\overline{t} = L^2/\kappa\overline{\rho}$.

c) When $x^2 > s(t)^2$, $\rho_F = 0$, and we immediately get

$$(\rho_F)_t = 0 = (\rho_F(\rho_F)_x)_x.$$

When $x^2 < s(t)$, $\rho_F = ct^{-\frac{1}{3}} - \frac{1}{6}x^2t^{-1}$, and we calculate

$$(\rho_F)_t = -\frac{1}{3}ct^{-\frac{4}{3}} + \frac{1}{6}x^2t^{-2}, \qquad (\rho_F)_x = -\frac{1}{3}xt^{-1}, \qquad (\rho_F)_{xx} = -\frac{1}{3}t^{-1},$$

to conclude that

$$\begin{aligned}(\rho_F(\rho_F)_x)_x &= ((\rho_F)_x)^2 + \rho_F(\rho_F)_{xx}\\ &= \frac{1}{9}x^2t^{-2} - \frac{1}{3}ct^{-\frac{4}{3}} + \frac{1}{18}x^2t^{-2}\\ &= -\frac{1}{3}ct^{-\frac{4}{3}} + \frac{1}{6}x^2t^{-2} = (\rho_F)_t.\end{aligned}$$

Now we will show that ρ_F solves the conservation law in integral form on any $0 < a < s(t) < b$: The time change of mass inside $[a, b]$ is equal the influx at $x = a$ (i.e. $+j(a, t)$) plus the influx at $x = b$ (i.e. $-j(b, t)$), and the (scaled) flux here is $j = -\rho\rho_x$ (see also the hint). Hence we have

$$\frac{d}{dt}\int_a^b \rho\,dx = -j(b,t) + j(a,t) = (\rho\rho_x)(b,t) - (\rho\rho_x)(a,t). \tag{6}$$

Since ρ_F is a (classical) solution for $x \in (a, s(t))$ and $\rho = 0$ for $x \in [s(t), b)$, we get

$$\begin{aligned}\frac{d}{dt}\int_{[a,b]}\rho_F\,dx &= \frac{d}{dt}\Big(\int_a^{s(t)}\rho_F\,dx + \underbrace{\int_{s(t)}^b \rho_F\,dx}_{=0}\Big)\\ &= \underbrace{\rho_F(s(t),t)s'(t)}_{=0} + \int_a^{s(t)}(\rho_F)_t\,dx \qquad \text{(Leibniz rule)}\\ &= -\int_a^{s(t)}(\rho_F(\rho_F)_x)_x\,dx\\ &= -\rho\rho_x|_a^{s(t)} = -\rho\rho_x|_a^b,\end{aligned}$$

and ρ_F satisfies (6).

Since $\rho_F = 0$ for $|x| > s(t)$, we first note that $\int_{-s(t)}^{s(t)} \rho_F \, dx = \int_{-\infty}^{\infty} \rho_F \, dx = 1$. Then since also $\rho_F \geq 0$ and $\lim_{t\to 0} s(t) = 0$, for any continuous function f,

$$\begin{aligned}\Big| \int_{-\infty}^{\infty} \rho_F(x,t) f(x)\, dx - f(0) \Big| &= \Big| \int_{-s(t)}^{s(t)} \rho_F(x,t) \big(f(x) - f(0)\big)\, dx \Big| \\ &= \max_{|x|\leq s(t)} |f(x) - f(0)| \int_{-s(t)}^{s(t)} \rho_F(x,t)\, dx \\ &= \max_{|x|\leq s(t)} |f(x) - f(0)| \to 0 \qquad \text{as} \qquad t \to 0.\end{aligned}$$

The function ρ_F is a fundamental solution of equation (5) on the exam, if it is a solution with initial condition equal to the delta function. The meaning of the initial condition is that $\lim_{t\to 0} \int_{-\infty}^{\infty} \rho_F(x,t) f(x) = f(0)$ for all continuous functions f. Both requirements are satisfied by the previous parts of problem **c)**, and hence ρ_F is a fundamental solution.

d) The idea of the method of intermediate asymptotics is to rescale the problem for large (but finite!) times in such a way that the rescaled intitial data approximates the δ-function. Then the solution of the rescaled problem will be close to the fundamental solution of this problem. Going back to original variables, we get an approximation valid for large times.

Let $\bar{t} = 3\,\text{months}$ and $t_1 = 5\,\text{days}$. Here we want to rescale the initial value problem

$$\begin{cases} \rho^*_{t^*} + \kappa(\rho^* \rho^*_{x^*})_{x^*} = 0, \\ \rho^*(x^*, t_1) = \rho^*_0(x^*), \end{cases}$$

such that

(i) the time scale is $\bar{t} = 3\,\text{months}$ (large compared to t_1),

(ii) the scaled equation becomes equation (5) on the exam (to be able to use part **c)**),

(iii) the scaled initial data $\rho(x,0) \approx \delta(x)$, the δ-function.

That is, we seek a scaled problem of the form

$$\begin{cases} \rho_t + (\rho \rho_x)_x = 0, \\ \qquad \rho(x,0) \approx \delta(x), \end{cases} \tag{7}$$

If we can do that, we can conclude (by stability of the PDE) that the scaled solution $\rho \approx \rho_F$, where ρ_F is the fundamental solution given in part **c)**. Going back to original variables, we find an approximation of the solution ρ^* at $t^* = \bar{t} = 3\,\text{months}$.

Let us now show that (7) holds after scaling. Take $\rho^* = \overline{\rho}\rho$, $t^* = Tt$, and $x^* = Xx$. The time scale $T = \bar{t}$, and we choose X and $\overline{\rho}$ such that (7) holds.

From the equation, we find that

$$\rho^*_{t^*} + \kappa(\rho^* \rho^*_{x^*})_{x^*} = 0 \quad \implies \quad \frac{\overline{\rho}}{\bar{t}}\rho_t + \frac{\kappa\overline{\rho}^2}{X^2}(\rho\rho_x)_x = 0 \quad \implies \quad \rho_t + \frac{\kappa\overline{\rho}\bar{t}}{X^2}(\rho\rho_x)_x = 0,$$

and then we get (7) if

$$\frac{\kappa\overline{\rho}^2\bar{t}}{X^2} = 1.$$

Next we look at the initial condition. Recall that $\rho^*_0 = 0$ for $|x^*| > l := 5\,\mathrm{m}$ and of total mass $M := \int_{-\infty}^{\infty} \rho^*\, dx^*$. We want to choose a scaling such that $\int_{-\infty}^{\infty} \rho\, dx = 1$ and $\rho(x, 0) = 0$ for $|x| > \varepsilon$ for some $0 < \varepsilon \ll 1$, or in other words, $\rho(x, 0) \approx \delta(x)$. The integral 1 conditions is satisfied if

$$M = \int_{-\infty}^{\infty} \rho^*\, dx^* = \overline{\rho}X \int_{-\infty}^{\infty} \rho\, dx = \overline{\rho}X.$$

We now have two equations for two unknowns X and $\overline{\rho}$, and the solution is

$$\overline{\rho} = \left(\frac{M^2}{\kappa\bar{t}}\right)^{\frac{1}{3}}, \qquad X = \left(\kappa M\bar{t}\right)^{\frac{1}{3}}.$$

Note that $\rho(x, \frac{t_1}{\bar{t}}) = \frac{1}{\overline{\rho}}\rho^*(Xx, t_1) = 0$ for $|Xx| \geq l$ or $|x| \geq \frac{l}{X}$. Since $\bar{t} = 3\text{months} \simeq 8 \cdot 10^6\text{s}$, a quick calculation shows that

$$X = \left(\kappa MT\right)^{\frac{1}{3}} \simeq \left(10^{-5} \cdot 10^8 \cdot 8 \cdot 10^6\right)^{\frac{1}{3}} \text{ m} = 2000 \text{ m} \gg 5 \text{ m} = l.$$

Hence $\frac{l}{X} \ll 1$, and since also $\frac{t_1}{\bar{t}} \ll 1$, it follows that

$$\rho(x, 0) \simeq \rho(x, \frac{t_1}{\bar{t}}) \simeq \delta(x).$$

We conclude that an approximation of ρ^* at $t^* = \bar{t} = 3$ months, is given by

$$\rho^*(x^*, \bar{t}) \simeq \overline{\rho}\rho_F\left(\frac{x^*}{X}, \frac{t^*}{\bar{t}}\right) = \left(\frac{M^2}{\kappa\bar{t}}\right)^{\frac{1}{3}} \rho_F\left(\frac{x^*}{(\kappa M\bar{t})^{\frac{1}{3}}}, 1\right).$$

As the rock formation is surrounded by impermeable rock, there can be no flux at the boundaries, $x^* = \pm\frac{L}{2}$. Thus the boundary conditions are

$$0 = j^*(\pm\tfrac{L}{2}, T) = \rho^*(\pm\tfrac{L}{2}, T)\rho^*_{x^*}(\pm\tfrac{L}{2}, T).$$

This is indeed satisfied by our approximation since

$$\frac{\frac{L}{2}}{X} = \frac{\frac{L}{2}}{(\kappa M\bar{t})^{\frac{1}{3}}} = 2.5 > 1.651 \simeq (6c)^{\frac{1}{2}} = s(1),$$

and then since $\rho_F(x,t) = 0$ for $|x| > s(t)$,

$$\rho^*(\pm\frac{L}{2}, \bar{t}) = \bar{\rho}\rho_F\left(\frac{\frac{L}{2}}{X}, 1\right) = 0.$$