



1 Let

$$(1) \quad u(x) = e^{-10x} + e^{-100x} \quad \text{for } x \in [0, 1].$$

Suggest (natural) scales for x and indicate where in $[0, 1]$ their use is reasonable.

2 (Problem 7 p. 32 in Logan)

A rocket blasts off from the earth's surface. During the initial phase of flight, fuel is burned at the maximum possible rate α , and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations:

$$(2) \quad m'(t) = -\alpha, \quad m(0) = M,$$

$$(3) \quad v'(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{\left(1 + \frac{x(t)}{R}\right)^2}, \quad v(0) = 0,$$

$$(4) \quad x'(t) = v(t), \quad x(0) = 0$$

where $m(t)$ is the mass of the rocket, $v(t)$ is the upward velocity, $x(t)$ is the height above the earth's surface, M is the initial mass, g is the gravitational constant, and R is the radius of the earth. Reformulate the problem in terms of dimensionless variables using appropriate scales for m , x , v , t .

(Hint: Scale m and x by obvious choices; then choose the time scale and velocity scale to ensure that the terms in the v equation are of the same order as well as the terms in the x equation. Assume that the acceleration is due primarily to fuel burning and that the gravitational force is small in comparison.)

3 (Problem 4.2.6 p. 55 in Krogstad)

Consider the problem

$$(5) \quad \begin{aligned} y''(t) + \varepsilon y'(t) + 1 &= 0, \\ y(0) &= 0, \quad y'(0) = 0, \quad 0 < \varepsilon \ll 1. \end{aligned}$$

Determine the start of the perturbation expansion $y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t)$ to the solution for $t \geq 0$. Compare to the exact solution. (Hint: The general solution of (5) has the form $y(t) = A + B e^{-\varepsilon t} - \frac{t}{\varepsilon}$)

4 (Problem 4.2.7 p. 55 in Krogstad)

This problem is somewhat similar to the sinking object in a fluid that has been discussed in the lecture. However, we ignore now gravity (or assume that we are in a situation where it is much smaller than the other forces) but instead assume that the friction is non-linear. In this case, a possible model for the velocity reads as

$$(6) \quad m \frac{dv^*}{dt^*} = -av^* + b(v^*)^2,$$

with initial velocity

$$(7) \quad v^*(0) = V_0.$$

Here told that $a, b > 0$, and we assume that $bV_0 \ll a$ (that is, the linear part of the friction is dominant).

(a) Find the (obvious) scale for v^* and then the scale for time, T , from the simplified equation $m \frac{dv^*}{dt^*} = -av^*$ and the "rule of thumb"

$$T = \frac{\max |v^*|}{\max |dv^*/dt^*|}.$$

Show that this scaling leads to the equation

$$(8) \quad \frac{dv}{dt} = -v + \varepsilon v^2, \quad v(0) = 1, \quad \varepsilon \ll 1.$$

(b) Determine v_0 and v_1 of the series expansion $v(t) = v_0(t) + \varepsilon v_1(t) + \dots$. Is this result reasonable for all $t > 0$?

Note that the general solution of $\dot{y} = -y + \varepsilon y^2 = 0$ is

$$y(t) = \frac{e^{-t}}{C + \varepsilon e^{-t}},$$

where C is a constant?