Norwegian University of Science and Technology
Department of Mathematical Sciences

Solutions to exercise set 2

$\boxed{1}$ In our notation we write

$$u(x^*) = \mathrm{e}^{-10x^*} + \mathrm{e}^{-100x^*}, \quad x^* \in [0,1]$$

for the unscaled relation.

We observe that the function u follows three different behaviours within the interval $[0,1]$: At first, both terms are relevant, although the second term decreases much faster than the first one (or: the derivative of the second term dominates); then, the second term is essentially negligible, while the first one is still much larger than zero; finally, both of the terms are essentially equal to zero.

Natural scalings:

$$\begin{aligned}
&1.) \quad x^* = \frac{1}{100}x, && \text{then } \mathrm{e}^{-100x^*} \sim 1, \text{ when } x \sim 1;\\
&2.) \quad x^* = \frac{1}{10}x, && (\text{then } \mathrm{e}^{-10x^*} \sim 1 \text{ and } \mathrm{e}^{-100x^*} \ll 1, \text{ when } x \sim 1;\\
&3.) \quad x^* = 1 \cdot x, && (\text{then } u \sim 0, \text{ when } x \sim 1.
\end{aligned}$$

Reasonable regions for these scalings are as follows:

1. For the first scaling $x^* = \frac{1}{100}x$, we can use values $x \in [0,2]$, corresponding to $x^* \in \left[0, \frac{2}{200}\right]$. Then

$$u(x) = \mathrm{e}^{-\frac{1}{10}x_1} + \mathrm{e}^{-x_1} \approx 1 + \mathrm{e}^{-x_1}$$

or

$$u(x^*) \approx 1 + \mathrm{e}^{-100x^*}$$

2. Here we might choose values $x \in \left[\frac{2}{10}, 2\right]$, corresponding to $x^* \in \left[\frac{2}{100}, \frac{2}{10}\right]$. Then

$$u(x) = \mathrm{e}^{-x} + \mathrm{e}^{-10x} \approx \mathrm{e}^{-x}$$

or

$$u(x^*) \approx \mathrm{e}^{-10x^*}.$$

3. Here we can choose $x = x^* \in \left[\frac{2}{10}, 1\right]$, which yields

$$u(x) = \mathrm{e}^{-10x} + \mathrm{e}^{-100x} \approx 0.$$

In all cases, the boundaries between the regions should be understood to be fuzzy, and a small shift of them is perfectly fine.

$\boxed{2}$ In our notation the problem can be written as

$$m^{*\prime}(t^*) = -\alpha, \quad m^*(0) = M, \tag{1}$$

$$v^{*\prime}(t^*) = \frac{\alpha\beta}{m^*(t^*)} - \frac{g}{\left(1 + \frac{x^*(t^*)}{R}\right)^2}, \quad v^*(0) = 0, \tag{2}$$

$$x^{*\prime}(t^*) = v^*(t^*), \quad x^*(0) = 0. \tag{3}$$

Natural scalings for x^* and m^* are

$$x^* = Rx,$$
$$m^* = Mm.$$

Assuming acceleration is mainly due to the rocket engine, we neglect for the time being the gravity term. Then the scales V and T for velocity and time should be chosen in such a way that the remaining terms in (2) balance, that is,

$$\frac{V}{T} \sim \frac{V}{T} v'(t) = \frac{\alpha\beta}{M}.$$

Also, the terms in (3) should be well scaled, that is, in the equation

$$\frac{R}{T} x' = Vv,$$

the terms x' and v should be of the same order, implying that

$$R = VT.$$

Combining these relations yields the scalings

$$T = \sqrt{\frac{RM}{\alpha\beta}} \qquad \text{and} \qquad V = \sqrt{\frac{R\alpha\beta}{M}}.$$

In total, we obtain

$$x^* = Rx, \quad m^* = Mm, \quad t^* = \sqrt{\frac{RM}{\alpha\beta}}\, t, \quad v^* = \sqrt{\frac{R\alpha\beta}{M}}\, v.$$

Moreover, we obtain the scaled equations

$$\begin{aligned} m'(t) &= -\mu, \quad m(0) = 1, \\ v'(t) &= \frac{1}{m(t)} - \varepsilon \frac{1}{(1+x)^2}, \quad v(0) = 0, \\ x'(t) &= v(t), \quad x(0) = 0, \end{aligned}$$

with parameters

$$\mu = \sqrt{\frac{R\alpha}{M\beta}} \qquad \text{and} \qquad \varepsilon = \frac{Mg}{\alpha\beta}.$$

Here the parameter $\varepsilon = Mg/\alpha\beta$ will be small provided that our assumption that the acceleration is mainly due to the rocket engine was correct.

$\boxed{3}$ The exact solution of the initial value problem is $y_{exact}(t) = \frac{1}{\epsilon^2}\left(1 - e^{-\epsilon t}\right) - \frac{t}{\epsilon}$. Let $y_{approx}(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)$. We want y_{approx} to satisfy the initial data for every value of $0 < \epsilon \leq 1$. This implies that $y_0(0) = y_1(0) = y_2(0) = 0$ and $y_0'(0) = y_1'(0) = y_2'(0) = 0$. Inserting y_{approx} in the equation, we get

$$y_0'' + 1 + \epsilon\left(y_1'' + y_0'\right) + \epsilon^2\left(y_2'' + y_1'\right) = 0.$$

As this should hold for all $0 < \epsilon \leq 1$, we have

$$\begin{aligned} y_0'' + 1 &= 0, \\ y_1'' + y_0' &= 0, \\ y_2'' + y_1' &= 0. \end{aligned}$$

The initial values and integration of the equations gives

$$y_{approx}(t) = -\frac{1}{2}t^2 + \epsilon\frac{1}{6}t^3 - \epsilon^2\frac{1}{24}t^4.$$

The Taylor expansion of y_{exact} is given by

$$y_{exact}(t) = \frac{1}{\epsilon^2}\left(1 - \sum_{n=0}^{\infty}\frac{(-\epsilon t)^n}{n!}\right) - \frac{t}{\epsilon} = \sum_{n=2}^{\infty}\frac{\epsilon^{n-2}(-1)^{n+1}t^n}{n!}.$$

The difference between the exact and approximate solution is then

$$y_{exact} - y_{approx} = \sum_{n=5}^{\infty}\frac{\epsilon^{n-2}(-1)^{n+1}t^n}{n!} = t^2 O(\epsilon^3 t^3).$$

Thus, as long as t and ϵt are not too big, the approximation is very good. However, for large values of ϵt, the exact solution behaves like

$$y_{exact}(t) \sim -\frac{t}{\epsilon} = -\frac{1}{\epsilon^2}\epsilon t,$$

whereas the approximate solution behaves like

$$y_{approx}(t) \sim -\epsilon^2\frac{1}{24}t^4 = -\frac{1}{\epsilon^2}\frac{1}{24}\epsilon^4 t^4.$$

Thus, for large values of t, we only have a good approximation, if ϵ decreases at the same time.

$\boxed{4}$ (a) From the problem's nature we have $0 \leq v^*(t) \leq V_0$. Then, V_0 will be a scale for v^*, and moreover

$$\frac{\left|b(v^*)^2\right|}{|av^*|} \leq \frac{bV_0}{a} \ll 1.$$

We find a time scale from the simplified equation $m\frac{dv^*}{dt^*} + av^* = 0$ with solution $v^*(t^*) = A\exp\left(-\frac{a}{m}t^*\right)$, that is $T = \frac{m}{a}$. Alternatively, and this is easier, we find this scale by balancing the first and second term in equation (5) in the problem set (set $v^* = Vv$ and $t^* = Tt$):

$$m\frac{dv^*}{dt^*} \sim av^* \quad\Rightarrow\quad m\frac{V}{T}\frac{dv}{dt} \sim aVv \quad \underset{v,\dot{v}\sim 1}{\rightsquigarrow} \quad m\frac{V}{T} \sim aV \quad\Rightarrow\quad T \sim \frac{m}{a}.$$

Using this scaling, we obtain the equation in the desired form and $\varepsilon = bV_0/a \ll 1$.

(b) Plugging in $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$ into the equation, we get that

$$\begin{aligned} O\left(\varepsilon^0\right): &\quad \dot{v}_0 = -v_0, \\ O\left(\varepsilon^1\right): &\quad \dot{v}_1 = -v_1 + v_0^2. \end{aligned}$$

With the initial condition we obtain

$$\begin{aligned} v_0(t) &= \mathrm{e}^{-t}, \\ v_1(t) &= \mathrm{e}^{-t} - \mathrm{e}^{-2t}, \end{aligned}$$

or

$$v(t) = \mathrm{e}^{-t} + \varepsilon\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) + O\left(\varepsilon^2\right).$$

This is the so-called *regular perturbation*. We have seen previously that the approximative solution is not always reasonable when $t \to \infty$, and we thus need to check its long term validity.

From the theory we know that the exact solution has the form

$$v_{\mathrm{ex}}(t) = \frac{\mathrm{e}^{-t}}{1 - \varepsilon\left(1 - \mathrm{e}^{-t}\right)},$$

and since $0 \le 1 - \mathrm{e}^{-t} < 1$ for $t \ge 0$, we can write the solution as a convergent geometric series.

$$v_{\mathrm{ex}}(t) = \mathrm{e}^{-t} \sum_{k=0}^{\infty} \left(\varepsilon\left(1 - \mathrm{e}^{-t}\right)\right)^k$$

The initial terms in the perturbation expansion coincide with the initial terms in the series above, and we have:

$$v_{\mathrm{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right) = \mathrm{e}^{-t} \sum_{k=2}^{\infty} \left(\varepsilon\left(1 - \mathrm{e}^{-t}\right)\right)^k \le \mathrm{e}^{-t}\varepsilon^2 \sum_{m=0}^{\infty} \varepsilon^m = \frac{\varepsilon^2 \mathrm{e}^{-t}}{1 - \varepsilon} \le \frac{\varepsilon^2}{1 - \varepsilon}.$$

Thus, we have

$$\lim_{\epsilon \to 0} \left(\sup_{t > 0} \left| v_{\mathrm{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right) \right| \right) = 0,$$

and so $v_a(t) = v_0(t) + \varepsilon v_1(t)$ is a uniform approximation to the exact solution on the domain $t > 0$.