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Exercise set 5

1 (Exercise 2.1d p. 377 in Logan) Find the equilibrium points of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u^2(u^2 - 1)$$

and investigate their stability.

2 (Exercise 2.2 p. 377 in Logan) Determine the equilibrium points and sketch bifurcation diagrams for the following differential equations. Identify the bifurcation points and bifurcation solutions. Investigate the stability of the equilibrium points and indicate where a change of stability occurs.

(a)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (u - \mu)(u^2 - \mu)$$

(b)

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u(9 - \mu u)(\mu + 2u - u^2)$$

3 In a model of a chemical tube reactor, the concentration c of the reactant satisfies the following reaction-diffusion equation:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1 - c), \quad t > 0,\ x \in \mathbb{R}. \tag{1}$$

a) Show that the linearization about $c = 0$ of equation (1) is given by

$$\frac{\partial c_L}{\partial t} = \frac{\partial^2 c_L}{\partial x^2} + k c_L, \quad t > 0,\ x \in \mathbb{R}, \tag{2}$$

for some constant k. Determine k.

b) Assume in addition that we are given initial conditions

$$c_L(x, 0) = c_0(x), \quad x \in \mathbb{R}, \tag{3}$$

with $c_L \colon \mathbb{R} \to \mathbb{R}$ bounded and continuous. Show that the solution of (2)–(3) for any k is given by

$$\begin{aligned} c_L(x,t) &= e^{kt}(c_0 * c_F)(x,t) \\ &= e^{kt} \int_{-\infty}^{\infty} c_0(y) \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \, dy. \end{aligned}$$

Hint: Use that $c_F(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ satisfies

$$\frac{\partial c_F}{\partial t} = \frac{\partial^2 c_F}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R}.$$

c) Show that

$$|c_L(x,t) - 0| \le e^{kt} \max_{x\in\mathbb{R}} |c_0(x) - 0|.$$

Hint: Use that $\int_{-\infty}^{\infty} c_F(x,t)dx = 1$.

Finally, we want to perform a (linear) stability analysis of the equation. That is, we consider small perturbations of the equilibrium point, and test whether the solutions of the linearised equation with the perturbation as initial value converge back to the equilibrium point as $t \to \infty$ (in which case the point is asymptotically stable).

Because we are dealing here with time-dependent functions $c(\cdot, t)$, this notion of stability depends on our definition of "small perturbations", that is, the choice of the norm on the space of the perturbations we consider. One natural possibility is to use the supremum norm $\|c_0\|_\infty := \sup_{x\in\mathbb{R}} |c_0(x)|$ and call a perturbation small if it has a small maximal amplitude. However, choosing the L^2-norm $\|c_0\|_2 := \left(\int_{\mathbb{R}} c_0(x)^2\, dx\right)^{1/2}$ also makes sense in many applications, as this choice corresponds (often) to perturbations of small total energy. In the following, we will choose the supremum norm, though.

d) Find all (constant) equilibrium points c_E of the equation (1). Determine whether they are stable or not according to linear stability analysis with respect to the supremum norm.

That is, given an equilibrium point c_E, determine first the linearisation of the equation (1) around c_E. Given an initial function c_0 with $\max_{x\in\mathbb{R}} |c_0(x)| < \infty$, determine the corresponding solution $\hat{c}$ of the linearised equation.

- The point c_E is asymptotically stable according to linear stability analysis, if $\lim_{t\to\infty} \max_{x\in\mathbb{R}} |\hat{c}(x,t)| = 0$ for every initial function c_0.
- Conversely, it is unstable according to linear stability analysis, if there exists an initial function c_0 such that $\max_{x\in\mathbb{R}} |\hat{c}(x,t)| \to \infty$ as $t \to \infty$.