Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195
Mathematical Modelling
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Exercise set 7

$\boxed{1}$ The non-linear PDE

$$u_t + uu_x = 0$$

is commonly known as *Burgers' equation*. It can be seen as a differential form of a conservation law with flux density $j(u) = \frac{1}{2}u^2$ and production rate $q = 0$.

(a) Compute solutions of Burgers' equation for the initial conditions

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

(b) We now consider the non-homogeneous Burgers equation with right hand side ("production term") $-u$, that is,

$$u_t + uu_x = -u.$$

Compute the solutions of this equation with initial condition

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x > 0. \end{cases}$$

$\boxed{2}$ Use the method of characteristics to solve these initial value problems for $t > 0$.

(a) $u_t + u_x = -u, \qquad u(x,0) = u_0(x)$

(b) $u_t + u_x = x, \qquad u(x,0) = u_0(x)$

$\boxed{3}$ (Problem 1.13 in Ockendon, Howison, Lacey, Movchan, Applied Partial Differential Equations)

Paint flowing down a wall has thickness $u(x,t)$ satisfying $\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0 \quad \text{for } t > 0$.

(a) Show that the characteristics are straight and that the Rankine–Hugoniot condition at a shock $x = S(t)$ is $\mathrm{d}S/\mathrm{d}t = \left[\frac{1}{3}u^3\right]_-^+ / [u]_-^+$.

A stripe of paint is applied at $t = 0$ so that

$$u(x,0) = \begin{cases} 0, & x < 0 \text{ or } x > 1, \\ 1, & 0 < x < 1. \end{cases}$$

(b) Show that, for small enough t,

$$u = \begin{cases} 0, & x < 0, \\ (x/t)^{1/2}, & 0 < x < t, \\ 1, & t < x < S(t), \\ 0, & S(t) < x, \end{cases}$$

where the shock is given by $x = S(t) = 1 + t/3$.

(c) Explain why this solution changes at $t = 3/2$, and show that thereafter

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{S}{3t}.$$

$\boxed{4}$ (Exercise 7.2.2 p. 73 in Krogstad)
A groundwater reservoir R, consists of porous sandstone filled with water. The pores constitute a constant fraction Φ of the volume ($0 < \Phi < 1$). The boundary to R, ∂R, is partially open so that water can flow in and out. Two wells are drilled in the reservoir.

(a) Set up a conservation law in the integral form of the water in the reservoir, where the flux $\mathbf{j}(\mathbf{x}, t)$ is not yet specified. What is the expression if the density of water ρ and all the properties of R are constant?

To find an expression for the flux of water $\left[\mathrm{m}^3/(\mathrm{m}^2 \cdot \mathrm{s})\right]$, we assume that it only depends on viscosity μ $[\mathrm{kg}/(\mathrm{m} \cdot \mathrm{s})]$, the fluid resistance in the rock (permeability K $\left[\mathrm{m}^2\right]$), and the pressure gradient ∇p. (Here and in the rest of the exercise, we assume that ρ is constant, so that we measure the water flow in volume.)

(b) Show by using dimensional analysis that

$$\mathbf{j} = -k \frac{K}{\mu} \nabla p,$$

where k is a dimensionless constant (the sign is physically reasonable).

To study how oil pollution in the reservoir will spread, we look at a rod-shaped rock sample, which in addition to water also contains oil. All the pores are either filled with water or oil, and a rock volume V will contain a volume $S_o \Phi V$ of oil and $S_v \Phi V$ of water, where $S_o + S_v = 1$. We assume that water and oil have the same pressure and that the flux (in the x-direction along the rod) can be written in the form

$$j_i = -k_i(S_i) \frac{K}{\mu_i} \frac{\partial p}{\partial x}, \quad i = o, v.$$

(c) Specify the conservation laws for oil and water for the section of the rod between $x = a$ and $x = b$, and show that if the pressure gradient is

$$q = j_o + j_v = \text{constant},$$

then, when $S \equiv S_v$, we get the hyperbolic equation

$$\Phi \frac{\partial S}{\partial t} + \frac{\partial}{\partial x} f(S) = 0,$$

$$f(S) = \frac{q k_v(S)/\mu_v}{k_o(1 - S)/\mu_o + k_v(S)/\mu_v}. \tag{1}$$

(d) Let $\mu_o = \mu_v$ and $k_o\,(1-S) = 1 - S^2,\ k_v(S) = S^2$. Solve (1) for $t > 0$ for a rod with length L when

$$\begin{aligned} S(x,0) &= 1 - x/L, \quad 0 \le x \le L, \\ S(0,t) &= 1, \quad 0 \le t. \end{aligned}$$