TMA4195
Mathematical Modelling
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Solutions to exercise set 8

$\boxed{1}$ a) The model starts by defining a density ρ^* which lies between 0 and a maximal density $\rho_{\max}$. The cars' speed is assumed to be a linear function of ρ^*, $v^* = v_{\max}\left(1 - \frac{\rho^*}{\rho_{\max}}\right)$. Thus, the flux becomes $j^* = \rho^* v^* = \rho^* v_{\max}\left(1 - \frac{\rho^*}{\rho_{\max}}\right)$. The scaling

$$\begin{aligned} \rho^* &= \rho\rho_{\max}, \\ x^* &= Lx, \\ t^* &= (L/v_{\max})t \end{aligned}$$

gives $j = \rho\,(1-\rho)$ and the equation

$$\begin{aligned} \rho_t + c\,(\rho)\,\rho_x &= 0, \\ c\,(\rho) &= \frac{dj}{d\rho} = 1 - 2\rho. \end{aligned}$$

Since $c\,(\rho)$ decreases when ρ increases, a situtation where $\rho\,(x_1,t) < \rho\,(x_2,t)$ will, for $x_1 < x_2$, develop a shock.

b) In $x = 0$ og $x = 1$ there are no possibilities for cars to accumulate. <u>Thus, the flux has to be continuous in these points for</u> $t > 0$.
Between $x = 0$ and 1 the flux will have the form

$$\frac{1}{2}\rho\,(1-\rho)\,.$$

Using the hint, we assume that $\rho = 1/2$ so that the flux is 1/8 in this region <u>for all</u> $t \geq 0$. Then, it remains to find the solution for $x < 0$ and $x > 1$. In these regions, $j = 1/8$ corresponds to two possible densities, namely the solutions of

$$\frac{1}{8} = \rho\,(1-\rho)\,.$$

I.e.

$$\begin{aligned} \rho_+ &= \frac{1}{2} + \frac{1}{4}\sqrt{2}, \\ \rho_- &= \frac{1}{2} - \frac{1}{4}\sqrt{2}. \end{aligned}$$

The situation in the region $x > 1$ is sketched in figure 1.
"Limit characteristics" which start in $x = 1$ have equations

$$\begin{aligned} x &= 1 + t, \\ x &= 1 + (1 - 2\rho_-)t = 1 + \frac{1}{2}\sqrt{2}t. \end{aligned}$$

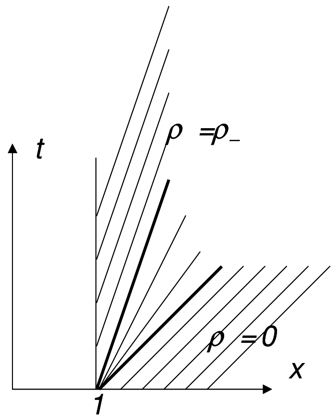

Figure 1: Characteristics for the solution in the region $x > 1$.

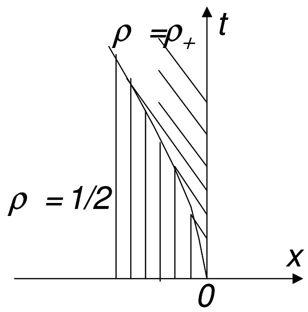

Figure 2: Sketch of the solutions to the left of $x = 0$.

In the region between these characteristics, we have a rarefaction wave such that

$$x = 1 + (1 - 2\rho)t,$$

i.e.

$$\rho(x,t) = \frac{1 - x + t}{2t}.$$

For $x < 0$ we have a situtation where $\rho = 1/2$ corresponds to a flux $\frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$. It is impossible to have such a flux immediately to the left side of $x = 0$, since the flux immediately to the right of $x = 0$ can be at most $1/8$. What is happening is that a queue is forming to the left of $x = 0$ with density $\rho = \rho_+$. In the back side of this queue we get a shock. The situation is sketched in figure 2.

We can deduce the shock's track from the differential equation

$$\frac{dx}{dt} = U = \frac{1/8 - 1/4}{\rho_+ - 1/2} = \frac{1/8 - 1/4}{\frac{1}{2} + \frac{1}{4}\sqrt{2} - 1/2} = -\frac{\sqrt{2}}{4}.$$

In other words, the shock has a constant speed and is governed by the equation

$$x = -\frac{\sqrt{2}}{4}t$$

which expresses a straight line starting in the origin.

The solution may be summed up as follows:

		ρ
$x<0$	$x<-\frac{\sqrt{2}}{4}t$	$\frac{1}{2}$
$x<0$	$x>-\frac{\sqrt{2}}{4}t$	$\frac{1}{2}+\frac{1}{4}\sqrt{2}$
$0<x<1$	$t\geq 0$	$\frac{1}{2}$
$1<x$	$t<x-1$	0
$1<x$	$x-1<t<(x-1)\sqrt{2}$	$\frac{1-x+t}{2t}$
$1<x$	$(x-1)\sqrt{2}<t$	$\frac{1}{2}-\frac{1}{4}\sqrt{2}$

$\boxed{2}$ (a) Let $z(t)=u(x(t),t)$ where $x(t)$ is given by $\dot{x}=z^{m-1}, x(t_0)=x_0$. The characteristic equations for the PDE then is

$$\begin{aligned}\dot{z}&=0,\\ \dot{x}&=z^{m-1}.\end{aligned}$$

Solving the system we obtain

$$\begin{aligned}z(t)&=z(t_0),\\ x(t)&=z(t_0)^{m-1}(t-t_0)+x_0.\end{aligned}$$

To determine $z(t_0), x(t_0)$ we use the initial and boundary conditions. The initial/boundary conditions gives

$$x(t)=\begin{cases}0+1\cdot(t-t_0), & t_0>0, x(t_0)=0,\\ x_0+t\cdot 0, & t_0=0, x(t_0)=x_0\in[0,1],\\ x_0+1\cdot t, & t_0=0, x(t_0)=x_0>1.\end{cases}$$

We will get a shock starting in the origin, and a rarefaction wave originating from $x=1$. The Rankine-Hugoniot condition on the shock gives speed $\dot{s}=\frac{\frac{1}{m}(u^+)^m-\frac{1}{m}(u^-)^m}{u^+-u^-}=\frac{1}{m}$, and thus the shock follows the line $x=s(t)=\frac{1}{m}t$. The rarefaction wave is found by letting $u(x,t)=\phi\left(\frac{x-1}{t}\right)$ and insert it into the PDE. This gives $\phi=\left(\frac{x-1}{t}\right)^{\frac{1}{m-1}}$ and we write the solution for $t\in[0,1]$ as follows

$$u(x,t)=\begin{cases}1, & x<\frac{1}{m}t,\\ 0, & \frac{1}{m}t<x<1,\\ \left(\frac{x-1}{t}\right)^{\frac{1}{m-1}}, & 1<x<1+t,\\ 1, & 1+t<x.\end{cases}$$

If $m<1$, the shock will eventually crash with the rarefaction wave, and the Rankine-Hugoniot condition for the shock will change.

(b) We have $f'(u)=u^2-\frac{4}{3}u-\frac{1}{3}$, and $f(u)=0$ implies $u\in\{-1,1,2\}$. Only characteristics going into the domain $x>1$ can affect the solution there and guarantee that the condition $f(u)=0$ holds. Two choices of u gives $f'(u)>0$, the first (i) is $u|_{x=1}=2$, the second (ii) is $u|_{x=1}=-1$.

(i) As in (a) we find the characteristic equations $\dot{z}=0, \dot{x}=f'(z)$ and solve. This gives $x(t)=x_0+(t-t_0)f'\big(u(x_0,t_0)\big)$. We insert the initial/boundary conditions

$$x(t)=\begin{cases}1+(t-t_0), & x_0=1, t_0>0,\\ x_0-\frac{1}{3}t, & x_0>1, t_0=0.\end{cases}$$

The characteristics collide and give a shock. The Rankine-Hugoniot gives the speed $\dot{s} = \frac{f(2)-f(0)}{2-0} = -\frac{1}{3}$, $s(t) = 1 - \frac{1}{3}t$. Which implies $u(1,t) = 2$, while $f(u(1,t)) = 2 \neq 0$ and the initial/boundary value problem does not have any solution.

(ii) Here the characteristics are given by the equation

$$x(t) = \begin{cases} 1 + 2(t - t_0), & x_0 = 1, t_0 > 0, \\ x_0 - \frac{1}{3}t, & x_0 > 1, t_0 = 0. \end{cases}$$

Again the characteristics collide to form a shock. The shock solution is given by

$$u(x,t) = \begin{cases} -1, & 0 < x < s(t) \\ 0, & s(t) < x, \end{cases}$$

where $s(t) = 1 + \frac{2}{3}t$ is determined by the Rankine-Hugoniot.

$\boxed{3}$ We will use the control volume $x \in [x_0, x_0 + \Delta x]$ at time t_0 to derive the conservation laws.

(a) There is no production of mass in the pipeline, this implies that the integral equation for conservation of mass reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_0}^{x_0+\Delta x} \rho(x, t_0) \, \mathrm{d}x = u(x_0, t_0)\rho(x_0, t_0) - u(x_0 + \Delta x, t_0)\rho(x_0 + \Delta x, t_0).$$

The only force acting on the gas is the pressure force. Thus conservation of momentum can be expressed

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_0}^{x_0+\Delta x} u(x, t_0)\rho(x, t_0) \, \mathrm{d}x + \rho(x_0 + \Delta x, t_0)u(x_0 + \Delta x, t_0)^2 - \rho(x_0, t_0)u(x_0, t_0)^2$$
$$= p(x_0) - p(x_0 + \Delta x).$$

(b) We find the differential form of the equations if we divide both equations by Δx and then let $\Delta x \to 0$. If we use the fact that $\lim_{\delta \to 0} \frac{1}{\delta} \int_y^{y+\delta} f(z) \, \mathrm{d}z = f(y)$, and assume that we can change the order of taking the limit and differentiating we get

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(u\rho) = 0,$$
$$\frac{\partial}{\partial t}(u\rho) + \frac{\partial}{\partial x}(u^2\rho + p) = 0.$$

The assumptions of ideal gas and constant temperature give

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(u\rho) = 0,$$
$$\frac{\partial}{\partial t}(u\rho) + \frac{\partial}{\partial x}(u^2\rho + \rho C T) = 0.$$

(c) If we are to include variable temperature we get one more unknown. So we need another equation. We have already used conservation of mass and momentum, so we have to use conservation of energy. We are given the energy density

$E = \frac{1}{2}\rho u^2 + \rho e$. The work done by the gas in the control volume on the surroundings is due to pressure. The first law of thermodynamics asserts that change in energy equals net flow of energy into the domain minus net work done by the system. This means that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_0}^{x_0+\Delta x} E(x,t_0)\,\mathrm{d}x =& u(x_0,t_0)E(x_0,t_0) - u(x_0+\Delta x,t_0)E(x_0+\Delta x,t_0) \\ &+ p(x_0,t_0)u(x_0,t_0) - p(x_0+\Delta x,t_0)u(x_0+\Delta x,t_0).\end{aligned}$$

All equations on differential form is then

$$\begin{aligned}\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(u\rho) &= 0, \\ \frac{\partial}{\partial t}(u\rho) + \frac{\partial}{\partial x}(u^2\rho + p) &= 0, \\ \frac{\partial}{\partial t}(E) + \frac{\partial}{\partial x}(uE + up) &= 0, \\ p &= \frac{E}{\rho} - \frac{1}{2}u^2.\end{aligned}$$

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(a) We may encounter rarefaction waves, contact discontinuities and shocks. For a treatment of these cases, we refer to Krogstad's note on Modeling Based on Conservation Principles, chapter 2.5.

(b) Figure 3 illustrates the situation - here, the inclined plane is rotated. The fluid in the control volume would accelerate due to gravity had it not been for the shear stress on the bottom of the control volume. The forces on the side edges cancel out.

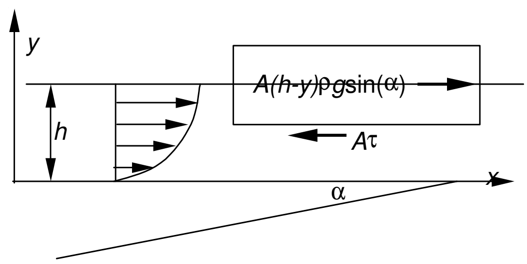

Figure 3: The fluid on the incline plane

Thereby, we get

$$\tau = (h-y)\rho g \sin\alpha,$$

or

$$\frac{du}{dy} = \frac{\rho g \sin\alpha}{\mu}(h-y).$$

This gives us $u(y) = \frac{\rho g \sin\alpha}{\mu} y(h - \frac{y}{2})$, since $u(0) = 0$. The total volume flux per unit of width becomes:

$$Q(h) = \int_0^h u(y)dy = \frac{\rho g \sin\alpha}{\mu}\left[h\frac{y^2}{2} - \frac{y^3}{6}\right]_0^h = \frac{\rho g \sin\alpha}{3\mu}h^3. \tag{1}$$

(c) The conservation law on integral form is:

$$\frac{d}{dt}\int_a^b h(x,t)dx + Q(h(b)) - Q(h(a)) = 0,$$

leading to

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial Q}{\partial x^*} = \frac{\partial h^*}{\partial t^*} + \frac{\rho g \sin\alpha}{\mu}\frac{\partial}{\partial x^*}\left(\frac{h^{*3}}{3}\right) = 0,$$

where we have temporarily written h^*, x^* and t^* for variables with dimension. By introducing scalings, $h^* = Hh$, $x^* = xX$ and $t^* = Tt$, we get

$$\frac{\partial h}{\partial t} + \frac{TH^2}{X}\frac{\rho g \sin\alpha}{\mu}\frac{\partial}{\partial x}\left(\frac{h^3}{3}\right) = 0. \tag{2}$$

The equation gets its desired form by choosing

$$T = \left(\frac{H^2}{X}\frac{\rho g \sin\alpha}{\mu}\right)^{-1}.$$

(d) We will solve equation in the first quadrant when

$$h(x,t) = \begin{cases} 0 & \text{for } x > 0,\ t = 0, \\ \sqrt{t} & \text{for } x = 0,\ t \geq 0. \end{cases} \tag{3}$$

The cinematic velocity $c(h) = \frac{dQ}{dh} = h^2$ is greater than or equal to zero, and the characteristic starting in (x_0, t_0) is given by

$$x = x_0 + c(h(x_0, t_0))(t - t_0).$$

The characteristics starting at the x axis are all vertical, while the characteristics from the t axis become more and more horizontal as t_0 increases. For a characteristic from the point $(0, t_0)$ on the t axis we get

$$x = c(h = \sqrt{t_0})(t - t_0) = t_0(t - t_0),$$

yielding $t_0^2 - tt_0 + x = 0$, i.e. $t_0 = \frac{t}{2} \pm \sqrt{\frac{t^2}{4} - x}$. In the area $x < t^2/4$ we have, in other words, characteristics crossing in every point.

(e) Due to the ambiguities mentioned above, it is stated in the problem which characteristics meet in the shock $s(t)$. In front of the shock, there are characteristics from the x axis, and behind the shock there are characteristics from the t axis. We set up the equation for the shock speed:

$$U(s,t) = \frac{Q(h_2) - Q(h_1)}{h_2 - h_1} = \frac{Q(h_1)}{h_1} = \frac{h_1^2}{3} = \frac{t_0}{3} = \frac{1}{6}\left(t + \sqrt{t^2 - 4s}\right).$$

The differential equation for the shock becomes as following:

$$\frac{ds}{dt} = \frac{1}{6}\left(t + \sqrt{t^2 - 4s}\right),$$

and we may easily verify that the given solution $s(t) = 5t^2/36$ works. Thereby, the solution (for $0 \leq x, t$) becomes

$$h(x,t) = \begin{cases} 0, & 5t^2/36 < x, \\ \left(\frac{t}{2} + \sqrt{\frac{t^2}{4} - x}\right)^{1/2}, & x < 5t^2/36. \end{cases}$$

The characteristics and the shock is sketched in figure 4.

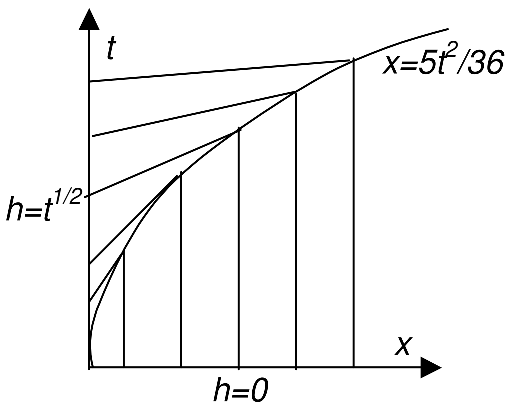

Figure 4: Sketch of the characteristics and the shock.