TMA4195
Mathematical Modelling
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Exercise set 1

$\boxed{1}$ A physical quantity F is dependent on the four (non-zero) physical quantities a, b, c, d. For two physical dimensions x and y, the dimensions of the physical quantities are:

$$[F] = x^2/y, \qquad [a] = y, \qquad [b] = 1/(xy), \qquad [c] = y/x^2, \qquad [d] = x/y,$$

a) Two valid conclusion of the Buckingham Pi Theorem applied to the above case, are given by

$$F = \frac{1}{c}\varphi\Big(a^2bd, acd^2\Big), \qquad\qquad F = \frac{d}{ab}\phi\Big(c^2d^3/b, ad\sqrt[3]{bc}\Big),$$

for functions φ and ϕ. Although seemingly different, prove that these conclusions are the same; that is, show both that for every ϕ, there is a φ, so the two expressions for F coincides, and vice versa.
Hint: For notational convenience, substitute $\pi_1 := a^2bd$, $\pi_2 := acd^2$, $\tilde{\pi}_1 := c^2d^3/b$ and $\tilde{\pi}_2 := ad\sqrt[3]{bc}$. To show the first part (for every ϕ, there is a φ) it suffices to show that the RHS of the equation $\varphi(\pi_1, \pi_2) = \frac{cd}{ab}\phi(\tilde{\pi}_1, \tilde{\pi}_2)$ can be written solely in terms of π_1 and π_2.

b) Suppose we learn that F also is dependent on the physical quantity e, with dimensions $[e] = x$. Consider the following expression

$$F = \frac{e^2}{a}\psi\Big(abe, \frac{ce^3}{a^2d}, \frac{a^5b^3d}{c}\Big),$$

for a function ψ. Why is this expression *not* a valid conclusion of the Buckingham Pi Theorem reapplied to cover the new information?

$\boxed{2}$ (Problem 4.1.3 p. 51 in Krogstad)
What are the possible choices of core variables (dimensional independent variables) from the set $\{R_1, \ldots, R_6\}$ if the dimension matrix looks like:

	R_1	R_2	R_3	R_4	R_5	R_6
F_1	1	1	-1	0	2	2
F_2	-2	-1	1	0	-3	-2
F_3	0	1	0	1	0	2

3 (Problem 4.1.11 p. 53 in Krogstad)
The necessary force (F) to keep a ship at a constant speed (U) depends on its shape, primarily the length (L), width (W), and its depth into the water (D). In addition, the water density, ρ, the water viscosity, ν, and the acceleration of gravity, g, play a part. Use dimensional analysis to find an expression for the force which includes the two most famous dimensionless numbers in ship design:

$$\text{Froude number: } Fr = U/\sqrt{Lg},$$
$$\text{Reynolds number: } Re = LU/\nu.$$

Ideally, a scale model[1] of the ship may be tested experimentally in water by keeping the dimensionless numbers for the model equal to those of the original ship. Is this really possible? Hints: $[F] = \mathrm{kgm/s^2}$, $[\rho] = \mathrm{kg/m^3}$, $[\nu] = \mathrm{m^2/s}$, $[g] = \mathrm{m/s^2}$.

4 We consider a rubber band which may be stretched to many times its original length l_0. The rubber band has a "density" ρ which we measure in mass per unit length, that is, kg/m. How does the density ρ vary when we stretch the band to a length l from its original length l_0 and density ρ_0?

After stretching the band more that twice its original length, we pluck the band like a guitar string. This experiments shows, somewhat unexpected, that the pitch (=frequency ω) remains almost constant when we vary the length (*try it*!). However, when stretching the band up towards its breaking limit, the frequency increases somewhat. The force F required for stretching the band to length l is proportional to $l - l_0$ over most of the range, that is, $F = F_0 \frac{l-l_0}{l_0}$, where F_0 is a constant.

Use dimensional analysis to explain the behaviour of the frequency.

Hint: Assume first that $\omega = f(l, \rho, F)$, apply dimensional analysis, and then introduce the expression of the density as a function of l, l_0 and ρ_0.

[1] A *scale model* is a model of the ship with the same geometric shape, but with a smaller size (say, $L =$1m for the model, compared to 100m for the original ship).