TMA4195
Mathematical Modelling
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Solutions to exercise set 1

$\boxed{1}$ a) Using the substitution in the hint, we see that

$$\tilde{\pi}_1 = \frac{\pi_2^2}{\pi_1}, \qquad \tilde{\pi}_2 = \sqrt[3]{\pi_1\pi_2},$$
$$\pi_1 = \frac{\tilde{\pi}_2^2}{\sqrt[3]{\tilde{\pi}_1}}, \qquad \pi_2 = \tilde{\pi}_2\sqrt[3]{\tilde{\pi}_1}.$$

We also note that

$$\frac{cd}{ab} = \frac{\pi_2}{\pi_1}, \qquad \frac{ab}{cd} = \frac{\tilde{\pi}_2}{\sqrt[3]{\tilde{\pi}_1}^2},$$

thus we can write

$$\varphi(\pi_1,\pi_2) = \frac{cd}{ab}\phi(\tilde{\pi}_1,\tilde{\pi}_2) = \frac{\pi_2}{\pi_1}\phi\Big(\frac{\pi_2^2}{\pi_1}, \sqrt[3]{\pi_1\pi_2}\Big),$$
$$\phi(\tilde{\pi}_1,\tilde{\pi}_2) = \frac{ab}{cd}\varphi(\pi_1,\pi_2) = \frac{\tilde{\pi}_2}{\sqrt[3]{\tilde{\pi}_1}^2}\varphi\Big(\frac{\tilde{\pi}_2^2}{\sqrt[3]{\tilde{\pi}_1}}, \tilde{\pi}_2\sqrt[3]{\tilde{\pi}_1}\Big).$$

b) The three dimensionless combinations of ψ are not independent:

$$\Big(abe\Big)^3 = \Big(\frac{ce^3}{a^2d}\Big)\Big(\frac{a^5b^3d}{c}\Big).$$

$\boxed{2}$ The rank of the dimension matrix is 3 and hence we can use as core variable any 3 R_i whose columns are independent.

Note that

$$2\overbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}^{R_2} = \overbrace{\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}}^{R_6}$$

so we remove R_6 for the time being. Then we note that only R_2 and R_4 contains dimension F_3 and hence one of these must be present in any choice of core variables. Let us then try

$$\begin{array}{llll}
R_2: & R_2R_1R_3 & \cancel{R_2R_3R_4} & R_2R_4R_5 \\
 & R_2R_1R_4 & R_2R_3R_5 & \\
 & R_2R_1R_5 & & \\
 & & & \\
R_4 \text{ but not } R_2: & R_3R_1R_4 & R_3R_4R_5 & \\
 & R_4R_1R_5 & &
\end{array}$$

It is easy to see that all these combinations have independent columns in the dimension matrix, except $R_2R_3R_4$.

Taking into account R_6, we find the same combinations as for R_2, but with R_6 replacing R_2. All in all, there are 13 possible choices of core variables.

3 Using the information provided in the problem, we assume

$$F = f\left(U, L, W, D, \rho, \nu, g\right).$$

The dimension matrix follows immediately and is shown in Table 1.

	F	U	L	W	D	ρ	ν	g
m	1	1	1	1	1	-3	2	1
s	-2	-1	0	0	0	0	-1	-2
kg	1	0	0	0	0	1	0	0

Table 1: Dimension matrix

The rank is 3, and there are several possibilities for core variables (avoiding F): (U, L, ρ), (g, D, ρ), (ν, ρ, W), ... However, if one aims for the Froude and Reynolds numbers, the choice (U, L, ρ) looks reasonable. With 8 variables, there are $8 - 3 = 5$ dimensionless combinations.

Since Re involves ν and Fr involves g, it is easy to arrive at the formula

$$F = \rho U^2 L^2 \times \Phi\left(\mathrm{Re}, \mathrm{Fr}, \frac{W}{L}, \frac{D}{L}\right).$$

If we want to use tests with a scale model in order to predict the behaviour of a real sized ship, we have to measure the function Φ for those dimensionless parameters that are typical for the original ship. Since the scale model keeps W/L and D/L (and more general, the *shape* of the ship) unchanged, it is sufficient to measure the function

$$\tilde{\Phi}(\mathrm{Re}, \mathrm{Fr})$$

for typical Froude and Reynolds numbers.

Now recall that the Froude number was $\mathrm{Fr} = U/\sqrt{Lg}$, and the Reynolds number was $\mathrm{Re} = LU/\nu$. If we thus scale the original ship down by a scaling factor of c, that is, we replace L by L/c, we have to scale the speed of the ship down by a factor of $\sqrt{c}$ in order to obtain the same Froude number (realistically, we cannot change the gravitational acceleration g). If we also want to keep the Reynolds number constant, this means that we have to scale down the viscosity of the fluid by a factor of $c^{3/2}$. Up to a certain degree (a factor 4–5), changes in viscosity of water can be achieved by heating, but this does not allow us to obtain reasonable scalings c. Alternatively, we could try to replace water by a fluid with lower viscosity, but, again, the necessary viscosities are not realistically achievable.

4 Let us begin setting up the dimension matrix for the physical quantities involved in the problem.

	ω	l	ρ	F
kg	0	0	1	1
m	0	1	-1	1
s	-1	0	0	-2

This matrix has rank 3. We easily find three linearly independent columns, for example 1, 2 and 3 and so we choose ω, l and ρ as core variables. The first dimensionless combination we find is $\pi_1 = F/(\rho^x l^y \omega^z)$. The unknowns can be found easily and are $x = 1$, $y = 2$ and $z = 2$, that is $\pi_1 = F/(\rho l^2 \omega^2)$. If there is relationship between these quantities, it has to be of the form $f(\pi_1) = 0$. Since the frequency ω is uniquely determined, it follows that π_1 is a constant.[1] This implies that

$$\omega = C\sqrt{\frac{F}{\rho l^2}}.$$

We now consider the situation where we are stretching a given rubber band and observe how the frequency changes. Since the mass of the rubber band does not change, if we stretch it, it follows that ρl is constant. Thus, for this particular situation we obtain that

$$\omega = \hat{C}\sqrt{\frac{F}{l}}.$$

Next, we assume (as discussed in the problem description) that the force F required for stretching the rubber band is proportional to the length change $l - l_0$, that is,

$$F \approx F_0(l - l_0),$$

at least as long as the force is well below the rupture point of the band. As a consequence, we obtain that

$$\omega = \hat{C}\sqrt{\frac{F}{l}} \approx \hat{C}\sqrt{\frac{F_0}{l_0}}\sqrt{1 - \frac{l_0}{l}} =: \omega_\infty\sqrt{1 - \frac{l_0}{l}}$$

for some frequency ω_∞ only depending on the original rubber band. Therefore, if we start with a rubber that is stretched to double its length, its frequency will rise from approximately $\omega_\infty/\sqrt{2}$ to ω_∞ as we stretch the band further and further. In total, the frequency will not change by more than a factor of $\sqrt{2}$, that is, its (musical) pitch by only *half an octave* (in equally tempered tuning, a frequency difference by a factor of $\sqrt{2}$ corresponds precisely to a diminished fifth or augmented fourth). Or, if we stretch a rubber band from twice its original length to four times its original length, its frequency will increase by a factor of $\sqrt{3/2}$ (which corresponds to a tonal interval somewhere between a minor and major third).

Finally, we note that this analysis breaks down near the rupture point of the rubber band: If we are close to the rupture point, the force required for stretching the band further is no longer proportional to the length change, but increases faster than the length. Thus we expect the frequency to rise again for a short time.

[1] Actually, this is only true up to a certain degree: If one solves the wave equation that describes the oscillations of this rubber band (by separation of variables and then a Fourier series ansatz to fit the initial conditions), it turns out that its oscillation can be decomposed as a (infinite) sum of oscillations of various frequencies that are all multiples of some basic frequency. If we say that we are only interested in that basic frequency, then the uniqueness argument is valid.