

- 1 Consider the problem

$$\ddot{x} + 2\varepsilon\dot{x} + \varepsilon x = 0, \quad x(0) = 0, \quad \dot{x}(0) = 1 - \varepsilon.$$

Determine x_0 and x_1 in the regular perturbation expansion

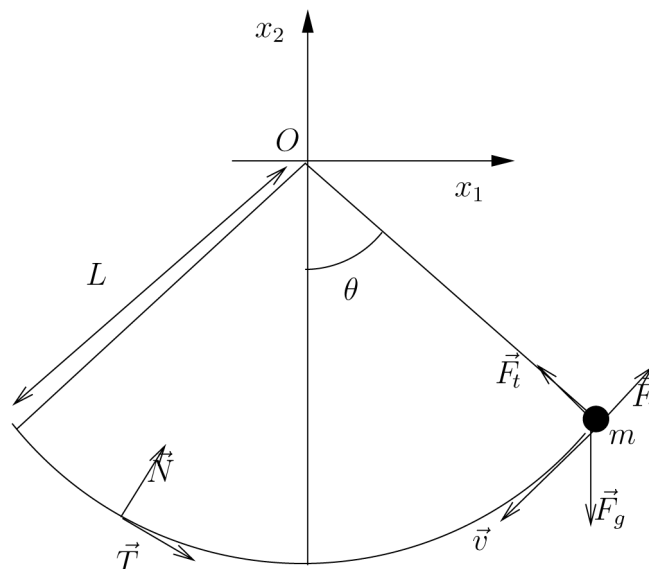
$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots.$$

- 2 (Problem 2, p. 23 in A. W. Bush: *Perturbation Methods for Engineers and Scientists*)

Obtain a two-term perturbation expansion for the solution of

$$\dot{y} - y - \varepsilon y^2 e^{-t} = 0, \quad y(0) = 1.$$

- 3 **Small angle approximation of a physical pendulum**



A pendulum of mass m and length L and point of fixation O is set in motion. The position \vec{x} of the pendulum is constrained to be on the circle

$$S = \{\vec{x} : |\vec{x}| = L\}.$$

As discussed in the lecture, its motion can be described, in the absence of friction,

by the equation

$$\begin{aligned} mL \frac{d^2 \vartheta^*}{d(t^*)^2} &= -g \sin(\vartheta^*), \\ \vartheta^*(0) &= \Theta_0, \\ \frac{d\vartheta^*}{dt^*}(0) &= 0. \end{aligned}$$

After rescaling, we can rewrite this as

$$\begin{aligned} \ddot{\vartheta} &= -\frac{1}{\varepsilon} \sin(\varepsilon \vartheta), \\ \vartheta(0) &= 1, \\ \dot{\vartheta}(0) &= 0, \end{aligned}$$

with small parameter $0 < \varepsilon \ll 1$. Now it is possible to apply the idea of asymptotic expansions and write the solution of this equation as

$$\vartheta = \vartheta_0 + \varepsilon \vartheta_1 + \varepsilon^2 \vartheta_2 + \dots$$

As we have seen in the lecture, the first terms in this expansion are

$$\begin{aligned} \vartheta_0(t) &= \cos t, \\ \vartheta_1(t) &= 0, \\ \vartheta_2(t) &= \frac{1}{192} (\cos t - \cos 3t) + \frac{1}{16} t \sin t. \end{aligned}$$

That is, we have

$$\vartheta(t) \approx \cos t + \frac{\varepsilon^2}{192} (\cos t - \cos 3t) + \frac{\varepsilon^2}{16} t \sin t$$

for small ε and t . The approximation, however, is not periodic, whereas the exact solution ϑ of the ODE is. The problem here is the term

$$\frac{\varepsilon^2}{16} t \sin t,$$

the amplitude of which grows with t . Such terms are called *secular* (in the sense of non-periodic, slowly changing with time).

In order to avoid secular terms in an asymptotic expansion, one can use the *Poincaré–Lindstedt method*. The idea here is to write

$$\vartheta(t) = \vartheta_0(\omega t) + \varepsilon \vartheta_1(\omega t) + \varepsilon^2 \vartheta_2(\omega t) + \dots$$

with

$$\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \dots$$

Then one tries to find functions $\vartheta_0, \vartheta_1, \vartheta_2, \dots$, and values $\omega_1, \omega_2, \dots$, in such a way that the secular terms are cancelled out in the approximations.

- a) Find equations for $\vartheta_0, \vartheta_1, \dots$, and show that

$$\vartheta_0(t) = \cos t.$$

- b) Determine ϑ_1 and ϑ_2 by choosing $\omega_1, \omega_2, \dots$ avoiding secular terms.