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Solutions to exercise set 3

$\boxed{1}$ Inserting $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$ into the equation and collecting terms of the same order of ε, we get

$$\ddot{x}_0 + \varepsilon\left(\ddot{x}_1 + 2\dot{x}_0 + x_0\right) + \varepsilon^2\left(\ddot{x}_2 + 2\dot{x}_1 + x_1\right) + \ldots = 0.$$

We hence get the following equations for x_0 and x_1:

$$\begin{aligned} \ddot{x}_0(t) &= 0, \\ \ddot{x}_1(t) &= -2\dot{x}_0(t) - x_0(t). \end{aligned}$$

For the initial conditions we obtain

$$\begin{aligned} x_0(0) &= 0, & \dot{x}_0(0) &= 1, \\ x_1(0) &= 0, & \dot{x}_1(0) &= -1. \end{aligned}$$

Note here that we have an inhomogeneous initial condition for $\dot{x}_1$, as the right hand side of the initial condition for the velocity $\dot{x}$ contains terms of order ε.

Solving now first for x_0, we obtain the solution $x_0(t) = t$. Inserted into the second equation, this leads to the equation

$$\ddot{x}_1(t) = -2 - t,$$

from which we get $x_1(t) = -\left(\frac{1}{6}t^3 + t^2 + t\right)$. Hence,

$$x(t) = t - \varepsilon\left(\frac{1}{6}t^3 + t^2 + t\right) + \mathcal{O}(\varepsilon^2).$$

$\boxed{2}$ We assume $y = y_0 + \varepsilon y_1 + \cdots$, which we insert into the equation before collecting the terms of the same order up to order ε, and get

$$(\dot{y}_0 - y_0) + \varepsilon\left(\dot{y}_1 - y_1 - y_0^2 \mathrm{e}^{-t}\right) + \mathcal{O}(\varepsilon^2) = 0.$$

We hence get the following equations for y_0 and y_1:

$$\begin{aligned} \dot{y}_0(t) - y_0(t) &= 0, \\ \dot{y}_1(t) - y_1(t) &= y_0^2(t)\mathrm{e}^{-t}. \end{aligned}$$

For the initial conditions, we have $y_0(0) = 1$ and $y_1(0) = 0$.

Solving first for y_0, we get $y_0(t) = \mathrm{e}^t$, which inserted into the second equation leads to

$$\dot{y}_1(t) - y_1(t) = \mathrm{e}^t.$$

Multiplying both sides of the equation with e^{-t}, we get

$$\frac{d}{dt}\left(\mathrm{e}^{-t}y_1(t)\right) = 1.$$

so that

$$\mathrm{e}^{-t}y_1(t) = t + C.$$

From the initial condition we find $C = 0$, and thus

$$y_1(t) = t\mathrm{e}^t.$$

Collecting everything, we obtain the approximation

$$y(t) = \mathrm{e}^t + \varepsilon t\mathrm{e}^t + \mathcal{O}(\varepsilon^2).$$

$\boxed{3}$ Inserting the ansatz for ϑ into the ODE, we obtain the equation

$$\omega^2(\ddot{\vartheta}_0 + \varepsilon\ddot{\vartheta}_1 + \cdots) = -\frac{1}{\varepsilon}\sin\big(\varepsilon(\vartheta_0 + \varepsilon\vartheta_1 + \cdots)\big).$$

Using a Taylor series expansion of sin and our assumption on ω, this leads to

$$\begin{aligned}(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots)^2(\ddot{\vartheta}_0 + \varepsilon\ddot{\vartheta}_1 + \varepsilon^2\ddot{\vartheta}_2 + \cdots)& \\ = -(\vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \cdots) &+ \frac{1}{6}\varepsilon^2(\vartheta_0 + \varepsilon\vartheta_1 + \cdots)^3 + \cdots,\end{aligned}$$

or

$$\begin{aligned}\ddot{\vartheta}_0 + \varepsilon(2\omega_1\ddot{\vartheta}_0 + \ddot{\vartheta}_1) + \varepsilon^2((2\omega_2 + \omega_1^2)\ddot{\vartheta}_0 + 2\omega_1\ddot{\vartheta}_1 + \ddot{\vartheta}_2) + \cdots& \\ = -\vartheta_0 - \varepsilon\vartheta_1 - \varepsilon^2&\left(\vartheta_2 - \frac{1}{6}\vartheta_0^3\right) - \cdots.\end{aligned}$$

For the initial conditions, we see that we only obtain homogeneous initial conditions apart for the conditions for ϑ_0. Thus we obtain the following equations:

$$\begin{aligned}\mathcal{O}(1):&\quad \ddot{\vartheta}_0 = -\vartheta_0; \quad \vartheta_0(0) = 1, \quad \dot{\vartheta}_0(0) = 0\\ \mathcal{O}(\varepsilon):&\quad 2\omega_1\ddot{\vartheta}_0 + \ddot{\vartheta}_1 = -\vartheta_1; \quad \vartheta_1(0) = 0, \quad \dot{\vartheta}_1(0) = 0\\ \mathcal{O}(\varepsilon^2):&\quad (2\omega_2 + \omega_1^2)\ddot{\vartheta}_0 + 2\omega_1\ddot{\vartheta}_1 + \ddot{\vartheta}_2 = -\vartheta_2 + \frac{1}{6}\vartheta_0^3; \quad \vartheta_2(0) = 0, \quad \dot{\vartheta}_2(0) = 0\\ &\quad \vdots\end{aligned}$$

From the first equation, we obtain that

$$\vartheta_0(t) = \cos t.$$

Inserting this into the second equation, we further obtain

$$\ddot{\vartheta}_1 + \vartheta_1 = 2\omega_1\cos t$$

with homogeneous initial conditions $\vartheta_1(0) = 0$ and $\dot{\vartheta}_1(0) = 0$. The solution of this equation is

$$\vartheta_1(t) = \omega_1 t\sin t.$$

Unless $\omega_1 = 0$, this is a secular term. As a consequence, we have to choose $\omega_1 = 0$ and $\vartheta_1 = 0$.

Next we continue with the equation for ϑ_2. We have

$$
\begin{aligned}
\ddot{\vartheta}_2 + \vartheta_2 &= \frac{1}{6}\vartheta_0^3 - (2\omega_2 + \omega_1^2)\ddot{\vartheta}_0 - 2\omega_1\ddot{\vartheta}_1 \\
&= \frac{1}{6}\cos^3 t + 2\omega_2 \cos t \\
&= \frac{1}{6}\frac{1}{4}(3\cos t + \cos 3t) + 2\omega_2 \cos t \\
&= \frac{1}{24}\cos 3t + \Big(\frac{3}{24} + 2\omega_2\Big)\cos t. \qquad (1)
\end{aligned}
$$

The homogeneous solution of this equation is $\cos t$. Similarly as for the determination of ω_1, we obtain that the only way of avoiding secular terms is by setting $\omega_2 = -\frac{1}{16}$; then the right hand side of the ODE simplifies to $\frac{1}{24}\cos 3t$, and we do not have any resonance with the homogeneous solution.

We thus obtain the equation

$$\ddot{\vartheta}_2 + \vartheta_2 = \frac{1}{24}\cos 3t$$

with homogeneous initial conditions. The solution of this equation is

$$\vartheta_2(t) = \frac{1}{192}\big(\cos t - \cos 3t\big).$$

We thus obtain the approximations

$$
\begin{aligned}
\vartheta(t) &= \vartheta_0(t) + \mathcal{O}(\varepsilon) \\
&= \cos(t) + \mathcal{O}(\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
\vartheta(t) &= \vartheta_0\Big(\Big(1 - \frac{\varepsilon^2}{16}\Big)t\Big) + \varepsilon^2\vartheta_2\Big(\Big(1 - \frac{\varepsilon^2}{16}\Big)t\Big) + \mathcal{O}(\varepsilon^3) \\
&= \cos\Big(\Big(1 - \frac{\varepsilon^2}{16}\Big)t\Big) + \frac{\varepsilon^2}{192}\Big[\cos\Big(\Big(1 - \frac{\varepsilon^2}{16}\Big)t\Big) - \cos\Big(3\Big(1 - \frac{\varepsilon^2}{16}\Big)t\Big)\Big] + \mathcal{O}(\varepsilon^3)
\end{aligned}
$$

for every $t > 0$ (actually, because of the symmetry with respect to ε, the approximation is of order $\mathcal{O}(\varepsilon^4)$). Note that there are no unbounded/secular terms anymore. Thus the error remains bounded for all $t > 0$.[1]

[1] We still cannot expect that the approximations converge uniformly on $\mathbb{R}_{\geq 0}$ to the actual solution as $\varepsilon \to 0$. Indeed, the difference between the periods of the actual solution and our approximation is $\omega - \omega_0 - \varepsilon^2\omega_2 = \mathcal{O}(\varepsilon^4)$ (the period ω is symmetric in ε and thus the third order term vanishes). Thus, at a time $t \sim 1/\varepsilon^4$, the true solution and the approximation will be off by half a period and the approximation error will be of size ~ 1. However, for times $t \ll 1/\varepsilon^4$ we can expect a very good approximation to the true solution. In contrast, with the "standard" asymptotic expansion we obtain the secular error term that increases with $t\varepsilon^2$. Thus the "standard" approximation is only useful for $t \ll 1/\varepsilon^2$.