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TMA4195 Mathematical Modelling Autumn 2019

Solutions to exercise set 4

1 We are going to study the boundary problem

$$\varepsilon y'' + (1+x)y' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1 \tag{1}$$

where $0 < \varepsilon \ll 1$. Let us introduce the notation $y = y_0 + \varepsilon y_1 + \mathcal{O}(\varepsilon^2)$ for the outer approximation and $Y = Y_0 + \varepsilon Y_1 + \mathcal{O}(\varepsilon^2)$ for the inner approximation. From the exercise we know that the boundary layer lies at $x = 0$. Let us call the uniform approximation y^u.

To find the outer approximation, we plug in the series expansion of y into (1) and find

$$(1+x)y_0' + y_0 + \varepsilon[y_0'' + (1+x)y_1' + y_1] = \mathcal{O}(\varepsilon^2).$$

We can fix y_0 using the equation $y_0'(1+x) + y_0 = 0$. This equation has general solution of the form $y_0 = C/(1+x)$. If we use the initial condition $y(0) = 0$ for fixing C, we get $y_0 \equiv 0$. With further computations we will see that this solution can not be used as the inner solution. Since the boundary layer lies at $x = 0$, we are going to use the initial condition at $x = 1$ for the outer approximation. Thus, asking that $y_0(1) = 1$, we get $C = 2$. The leading order for the outer approximation is

$$y(x) \approx y_0(x) = \frac{2}{1+x}.$$

To find the inner approximation Y, we scale x such that the new variable varies between 0 and 1 close to the boundary layer. Let us set $\xi = x/\varepsilon$ and obtain, using (1),

$$\frac{d^2Y}{d\xi^2} + \frac{dY}{d\xi} + \varepsilon(\xi\frac{dY}{d\xi} + Y) = 0. \tag{2}$$

Let us plug in the series expansion for Y into (2) and gather together the terms having same order. We get the following equation

$$\ddot{Y}_0 + \dot{Y}_0 + \varepsilon(\ddot{Y}_1 + Y_1 + \xi\dot{Y}_0 + Y_0) = \mathcal{O}(\varepsilon^2),$$

where $\dot{Y}$ means derivation with respect to ξ. We assume that this approximation is good near $\xi = 0$. We use the initial condition $Y(0) = 0$, that gives us $Y_0(0) = 0$. The initial term in the expansion of Y can be derived from the equation $\ddot{Y}_0 + \dot{Y}_0 = 0$. Using the initial condition $Y_0(0) = 0$, we get $Y_0 = K(1 - \mathrm{e}^{-\xi})$.

We have to match the inner and outer approximation. This can be done for Y_0 and y_0 by setting

$$\lim_{\xi\to\infty} Y(\xi) = \lim_{x\to 0} y(x),$$

which gives $K = 2$. The uniform approximation can be found summing together inner and outer and subtracting from their sum the common terms. This returns

$$y^u(x) = y(x) + Y(x/\varepsilon) - 2 = \frac{2}{1+x} - 2\mathrm{e}^{-x/\varepsilon}.$$

If we had used the initial condition $y(0) = 0$ instead of $y(1) = 1$, we would have found $y = 0$ and $K = 0 \Rightarrow Y = 0$. The uniform approximation would have been $y^u = 0$, which is clearly wrong. It is therefore right to use the initial condition $y(1) = 1$.

2 We begin by finding the outer solution. The boundary layer is given to be at $x = 0$. We neglect the $\epsilon y''$ term and obtain the boundary value problem for the outer solution

$$y_O' + y_O^2 = 0, \qquad y(1) = \frac{1}{2}.$$

The solution is $y_O(x) = \frac{1}{x+1}$.

To find the inner solution we scale $x = \delta\xi$ and let $Y(\xi) = y(x)$. The chain rule gives then

$$\frac{\epsilon}{\delta^2}Y'' + \frac{1}{\delta}Y' + Y^2 = 0, \qquad Y(0) = \frac{1}{4}.$$

We choose $\delta = \epsilon$ and multiply the equation by ϵ. This gives

$$Y'' + Y' + \epsilon Y^2 = 0.$$

The leading order solution can be found by solving $Y_I'' + Y_I' = 0$, $Y_I(0) = \frac{1}{4}$. The solution is $Y_I(\xi) = C_1 + C_2 e^{-\xi}$ with $C_1 + C_2 = \frac{1}{4}$. We use the matching requirement to find C_1 and C_2. To that end let $\Theta(\epsilon)$ be such that

$$\lim_{\epsilon\downarrow 0} \Theta(\epsilon) = 0, \qquad \lim_{\epsilon\downarrow 0} \frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty.$$

The matching requirement can be formulated

$$\lim_{\epsilon\downarrow 0} y_O\left(\eta\Theta(\epsilon)\right) = \lim_{\epsilon\downarrow 0} Y_I\left(\frac{\eta\Theta(\epsilon)}{\delta(\epsilon)}\right),$$

for all $\eta > 0$. This gives

$$C_1 + C_2 e^{-\infty} = 1,$$

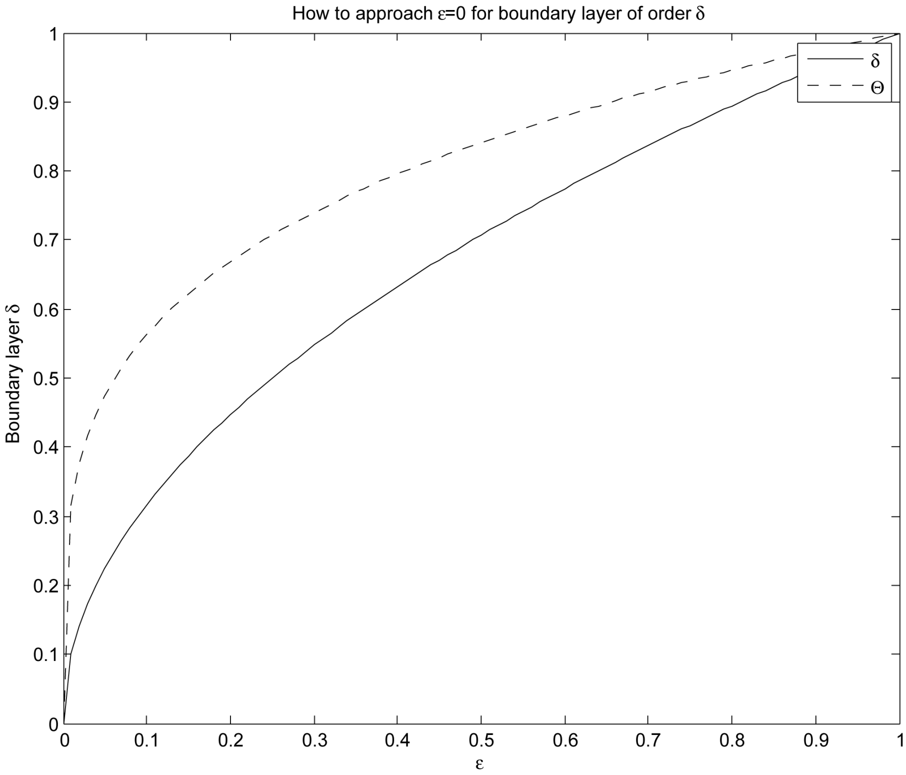
and thus $C_1 = 1, C_2 = -\frac{3}{4}$. The uniform approximation is then given by

$$\begin{aligned} y_u(x) &= y_O(x) + Y_I\left(\frac{x}{\delta}\right) - \lim_{\epsilon\downarrow 0} y_O\left(\eta\Theta(\epsilon)\right) \\ &= \frac{1}{x+1} - \frac{3}{4}e^{-\frac{x}{\epsilon}}. \end{aligned}$$

3 First we note that by the symmetry in the problem, u has to be an even function. To see this let $v(x) = u(-x)$ and observe that both v and u satisfies the equation. As u is even we have that $u'(-x) = -u'(x)$. Continuity of u' at 0 gives that $u'(0) = 0$ is the correct boundary condition.

By looking at the equation and boundary conditions we observe that for $x \approx 1$, $\epsilon u'' \approx -1$. Hence $|u''| \gg 1$ there, and if $u \sim \frac{1}{2}$ for x close to 0, u'' must be of order 1

Figure 1: The path Θ in the plane.



close to $x = 0$. This means that we should use a boundary layer around $x = 1$. First we find the outer solution u_O. We neglect the term $\epsilon u''$ in the equation and obtain

$$(2 - x^2)u_O = 1, \text{ or } u_O(x) = \frac{1}{2 - x^2}.$$

We see that $u'_O(0) = 0$. Now we turn to the inner solution. Rescale $x = 1 - \delta\xi$ and $U(\xi) = u(x)$. This implies $\frac{1}{\delta^2}U''(\xi) = u''(x)$, which gives the equation

$$\frac{\epsilon}{\delta^2}U'' - \left(2 - (1 - \delta\xi)^2\right) U = -1$$
$$\frac{\epsilon}{\delta^2}U'' - (1 + \mathcal{O}(\delta))\, U = -1.$$

We balance the terms by choosing $\delta = \epsilon^{\frac{1}{2}}$. The linear ODE has general solution $U(\xi) = c_1 e^{\xi} + c_2 e^{-\xi} + 1$. To determine c_1 and c_2 we need two boundary conditions. Observe first that $\xi = 0$ corresponds to $x = 1$. Thus $U(0) = u(1) = 0$, and the first equation is

$$c_1 + c_2 + 1 = 0.$$

We want the inner solution U and the outer solution u to match in a nice manner. That is, the transition from outside to inside the boundary layer should be continuous when $\epsilon \downarrow 0$. But as $\epsilon \downarrow 0$ the boundary layer shrinks as well. Let $\Theta(\epsilon)$ be a path in the plane such that $\Theta(\epsilon) \to 0$ as $\epsilon \downarrow 0$. The idea is to choose Θ such that for each $\epsilon > 0$ the function value $\Theta(\epsilon)$ lies in the intermediate region, at least for small ϵ. This happens if $\lim_{\epsilon\downarrow 0} \frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty$. See Figure 3. To sum up we have the following conditions on the function Θ

$$\lim_{\epsilon \downarrow 0} \Theta(\epsilon) = 0, \qquad \lim_{\epsilon \downarrow 0} \frac{\Theta(\epsilon)}{\delta(\epsilon)} = \infty.$$

The nice behaviour across the boundary layer can be formulated

$$\lim_{\epsilon \downarrow 0} u_O \left(1 - \eta\Theta(\epsilon)\right) = \lim_{\epsilon \downarrow 0} U \left(\frac{\eta\Theta(\epsilon)}{\delta(\epsilon)} \right),$$

for any parameter $\eta > 0$. In our case the left hand side will be equal to one. Thus

$$\lim_{\epsilon \downarrow 0} c_1 e^{\frac{\eta\Theta}{\sqrt{\epsilon}}} + c_2 e^{-\frac{\eta\Theta}{\sqrt{\epsilon}}} = 1 - 1 = 0.$$

This gives $c_1 = 0$ (or the exponential would go to infinity due to the limit of $\frac{\Theta}{\delta}$) and we get $c_2 = -1$. The uniform approximation is then given by

$$u_u(x) = \frac{1}{2 - x^2} + 1 - e^{-\frac{1-x}{\sqrt{\epsilon}}} - \lim_{\epsilon \downarrow 0} u_O(1 - \eta\Theta) = \frac{1}{2 - x^2} - e^{-\frac{1-x}{\sqrt{\epsilon}}}.$$