Norwegian University of Science and Technology
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TMA4195 Mathematical Modelling Autumn 2019

Solutions to exercise set 7

1 (a) The characteristic equations for Burgers' equation read

$$\dot{x} = z, \qquad x(0) = x_0,$$
$$\dot{z} = 0 \qquad z(0) = u_0(x_0).$$

Solving these equations gives

$$z(t) = z(0) = u_0(x_0)$$

and

$$x(t) = x_0 + t\,u_0(x_0).$$

Let us now first consider the case

$$u_0(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

In this case, the characteristics are

$$x(t) = x_0 + \begin{cases} 0 & \text{if } x < 0, \\ t & \text{if } x > t > 0. \end{cases}$$

The characteristics do not collide, and we have a well defined solution at all points covered by the characteristics. More precisely, the characteristics $x(t)$ starting at $x_0 < 0$ imply that

$$u(x(t), t) = u(x_0 + 0, t) = u_0(x_0) = 0,$$

while those starting at $x_0 > 0$ imply that

$$u(x(t), t) = u(x_0 + t, t) = u_0(x_0) = 1.$$

Or,

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x - t > 0. \end{cases}$$

However, there is a "dead zone" in the cone $0 < x < t$ that is not covered by characteristics Thus we model the solution in that region as a rarefaction wave

$$u(x,t) = \varphi\Big(\frac{x}{t}\Big) \qquad \text{if } 0 < x < t.$$

The function φ has to satisfy Burgers' equation, that is, we need that

$$0 = u_t + uu_x = -\frac{x}{t^2}\varphi'\Big(\frac{x}{t}\Big) + \frac{1}{t}\varphi\Big(\frac{x}{t}\Big)\varphi'\Big(\frac{x}{t}\Big)$$

or

$$0 = \frac{1}{t}\varphi'\Big(\frac{x}{t}\Big)\Big(-\frac{x}{t} + \varphi\Big(\frac{x}{t}\Big)\Big).$$

Thus

$$\varphi\Big(\frac{x}{t}\Big) = \frac{x}{t},$$

and we obtain the solution

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{x}{t} & \text{if } 0 < x < t, \\ 1 & \text{if } x > t > 0. \end{cases}$$

Now we consider the case

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

Here the characteristics are

$$x(t) = x_0 + \begin{cases} t & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

If either $x < 0$ or $x > t > 0$, there is a unique characteristic passing through the point (x,t). Thus we obtain that

$$u(x,t) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > t > 0. \end{cases}$$

However, in all points (x,t) with $t > x > 0$, two characteristics collide. Thus we model the solution as a shock solution with

$$u(x,t) = \begin{cases} 1 & \text{if } x < s(t), \\ 0 & \text{if } x > s(t), \end{cases}$$

where the shock s satisfies the Rankine–Hugoniot condition

$$\dot{s}(t) = \frac{[j]}{[u]} = \frac{j(u(s(t)^+,t)) - j(u(s(t)^-,t))}{u(s(t)^+) - u(s(t)^-)} = \frac{\frac{1}{2}1^2 - \frac{1}{2}0^2}{1-0} = \frac{1}{2}$$

with $s(0) = 0$. Thus

$$s(t) = \frac{1}{2}t,$$

and we obtain the shock solution

$$u(x,t) = \begin{cases} 1 & \text{if } x < \frac{1}{2}t, \\ 0 & \text{if } x > \frac{1}{2}t. \end{cases}$$

(b) The characteristic equations for this non-homogeneous equation are

$$\dot{x} = z, \qquad x(0) = x_0,$$
$$\dot{z} = -z, \qquad z(0) = \begin{cases} 0 & \text{if } x_0 < 0, \\ x_0 & \text{if } x_0 > 0. \end{cases}$$

For $x_0 < 0$, we immediately obtain the solutions $z(t) = 0$ and $x(t) = x_0$. For $x_0 > 0$, the equation for z has the solution

$$z(t) = u_0(x_0)e^{-t} = x_0e^{-t}.$$

Inserted in the equation for x, this yields

$$\dot{x} = x_0e^{-t}$$

and therefore, using the initial condition $x(0) = x_0$,

$$x(t) = x_0(2 - e^{-t}).$$

Since $1 < (2 - e^{-t}) \leq 2$ for all $t \geq 0$, it follows that the characteristics do not collide and cover the whole region $x > 0$, $t > 0$. Thus we obtain the solution

$$u\big(x(2 - e^{-t}), t\big) = xe^{-t}$$

for $x > 0$, $t > 0$. Or, setting $y = x(2 - e^{-t})$, we obtain

$$u(y, t) = \frac{ye^{-t}}{2 - e^{-t}} = \frac{y}{2e^t - 1}.$$

To summarise, we obtain

$$u(x, t) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{x}{2e^t - 1} & \text{if } x > 0. \end{cases}$$

2 (a) Let $x(t)$ be given by $\dot{x}(t) = 1, x(0) = x_0$ and define $z(t) = u\,(x(t), t)$. Then $\dot{z}(t) = u_t + u_x \cdot 1 = -u(x(t), t) = -z(t)$, and the characteristic equations are

$$\begin{aligned} \dot{x} &= 1, \qquad x(0) = x_0, \\ \dot{z} &= -z, \qquad z(0) = u_0(x_0). \end{aligned}$$

This gives $x = x_0 + t$ and $z(t) = u_0(x_0)e^{-t}$. But x_0 is not given, we want to compute $u(x, t)$. We calculate x_0 from $x(t)$, $x_0 = x - t$ and thus the solution is $u(x, t) = u_0(x - t)e^{-t}$.

(b) Let $x(t)$ be given by $\dot{x}(t) = 1, x(0) = x_0$ and define $z(t) = u\,(x(t), t)$. Then $\dot{z}(t) = u_t + u_x \cdot 1 = x(t)$, and the characteristic equations are

$$\begin{aligned} \dot{x} &= 1, \qquad x(0) = x_0, \\ \dot{z} &= x, \qquad z(0) = u_0(x_0). \end{aligned}$$

The solutions are $x(t) = x_0 + t, z(t) = z_0 + x_0t + \frac{1}{2}t^2$. This gives $u(x, t) = u_0(x - t) + xt - \frac{1}{2}t^2$.

Remark: For equations of the type $u_t + f(u, x)u_x = g(u, x)$ let $x(t) = f(z, x), z(t) = u(x(t), t)$. Then

$$\begin{aligned} \dot{z} &= u_t(x(t), t) + u_x(x(t), t)\dot{x}(t) \\ &= u_t + f(u, x)u_x = g(u, x) \\ &= g(z, x), \end{aligned}$$

and the characteristic equations are $\dot{x} = f(z, x), \dot{z} = g(z, x)$. We need to be able to solve the equations and find x_0, z_0 expressed with x, t, u_0.

3 (1.13 in Ockendon, Howison, Lacey, Movchan, Applied Partial Differential Equations)

a) Rankine-Hugoniot: for $u_t + \frac{\partial}{\partial x} f(u) = 0$:

$$\frac{\mathrm{d}S}{\mathrm{d}t}(u^+ - u^-) = f(u^+) - f(u^-)$$

Here $f(u) = \frac{1}{3}u^3$, so

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{3}\frac{(u^+)^3 - (u^-)^3}{u^+ - u^-}.$$

Characteristic equations: $z(t) = u(x(t), t)$

$$\begin{cases} \dot{x} = f'(z) = z^2, & x(0) = x_0 \\ \dot{z} = 0, & z(0) = u(x(0), 0) = u_0(x_0) \end{cases}$$

Solution:

$$\begin{cases} z = \text{const} = u_0(x_0) \\ x = x_0 + t(u_0(x_0))^2 \end{cases}$$

Hence characteristics are straight lines.

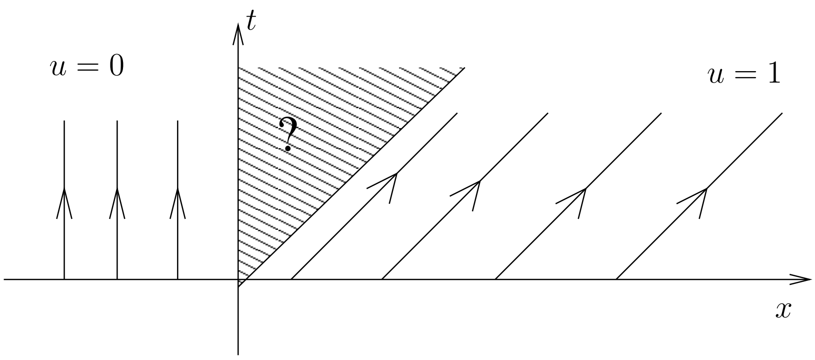
b) The problem consists of 2 Riemann problems:

$$u_t + u^2 u_x = 0, \quad u(x, 0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \quad (x < 1) \end{cases} \tag{1}$$

$$u_t + u^2 u_x = 0, \quad u(x, 0) = \begin{cases} 1 & x < 1 \quad (x > 0) \\ 0 & x > 1 \end{cases} \tag{2}$$

For (1):

$$x = x_0 + t \cdot \begin{cases} 0 & x_0 < 0 \\ 1 & x_0 > 0 \quad (x_0 < 1) \end{cases}$$



There is a "dead sector" and we know we have to fill it with a rarefaction fan:

$$u(x, t) = \phi\left(\frac{x}{t}\right) \tag{3}$$

i.e. a function constant at rays out of $(0,0)$. Insert (3) into (1) to find that

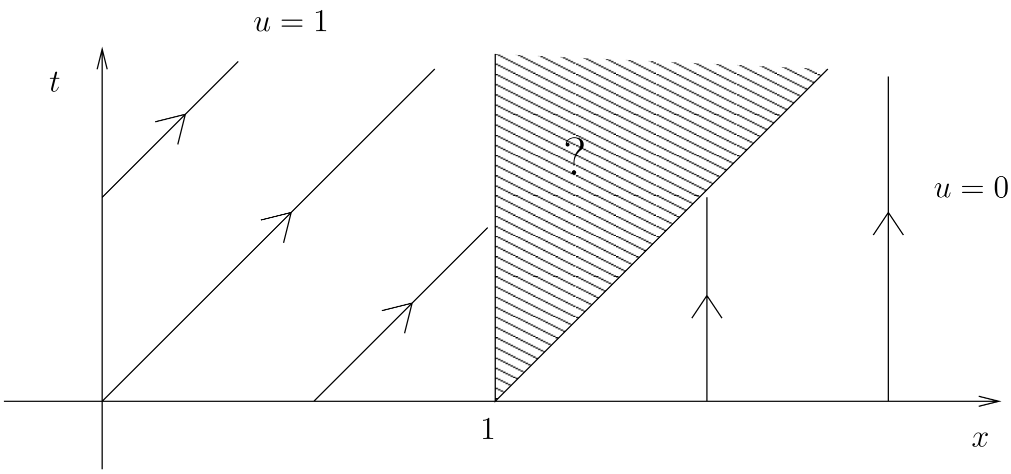
$$\phi'\left(\frac{x}{t}\right)\left(-\frac{x}{t^2}\right)+\phi^2\phi'\left(\frac{x}{t}\right)\frac{1}{t}=0$$
$$\Rightarrow \phi^2=\frac{x}{t}, \quad \Rightarrow \phi=\sqrt{\frac{x}{t}}$$

Solution:

$$u(x,t)=\begin{cases}0, & x<0\\ \sqrt{\frac{x}{t}}, & 0<x<t\\ 1, & x>t \quad \text{(but not too big)}\end{cases}$$

For (2):

$$x=x_0+t\cdot\begin{cases}1 & x_0<1 \quad (x_0>0)\\ 0 & x_0>1\end{cases}$$



Characteristics collide at $x=1$, hence there is a shock starting at $(x,t)=(1,0)$ and moving with speed

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{1}{3}\,\frac{0^3-1^3}{0-1}=\frac{1}{3}.$$

Hence

$$S(t)=1+\frac{1}{3}t$$

The shock solution then becomes

$$u(x,t)=\begin{cases}1 & x<S(t) \quad (x>t)\\ 0 & x>S(t)\end{cases}$$

Total solution

$$u(x,t)=\begin{cases}0 & x<0\\ \sqrt{\frac{x}{t}} & 0<x<t\\ 1 & t<x<1+\frac{1}{3}t\\ 0 & x>1+\frac{1}{3}t\end{cases}$$

Note: Solution only defined as long as $t\leq 1+\frac{1}{3}t$ or $t\leq\frac{3}{2}$.

c) For $t > \frac{3}{2}$, the left state of shock changes from (1) to $\sqrt{\frac{x}{t}} = \sqrt{\frac{S(t)}{t}} (< 0)$ and hence the speed of the shock changes to

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \frac{1}{3}\frac{(u^+)^3 - (u^-)^3}{u^+ - u^-} \\ &= \frac{1}{3}\frac{0^3 - \left(\frac{S(t)}{t}\right)^{3/2}}{0 - \sqrt{\frac{S(t)}{t}}} \\ &= \frac{1}{3}\frac{S(t)}{t}\end{aligned}$$

Since

$$S\left(t = \frac{3}{2}\right) = \frac{3}{2},$$

we solve (separable equations) and find that

$$S(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}} t^{1/3}$$

The new solution is then

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \sqrt{\frac{x}{t}}, & 0 < x < S(t) \\ 0, & S(t) < x \end{cases}$$

and this solution persists until $t = +\infty$.

4 a) From the general conservation law it follows immediately that

$$\frac{d}{dt}\int_R (\Phi\rho) dV + \int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma = \int_R q dV.$$

We assume that the wells are δ-functions (which is OK assuming the reservoir is large compared to the wells),

$$q(\mathbf{x}, t) = \sum_{n=1}^{2} q_n(t) \delta(\mathbf{x} - \mathbf{x}_n).$$

Thus

$$\frac{d}{dt}\int_R (\Phi\rho) dV + \int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma = q_1 + q_2.$$

If ρ is constant and the properties of R are unchanging in time, then

$$\frac{d}{dt}\int_R (\Phi\rho) dV = 0,$$

and the balance of the water is

$$\int_{\partial R} \mathbf{j} \cdot \mathbf{n} d\sigma = q_1 + q_2.$$

b) When

$$[p] = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\frac{1}{\text{m}^2} = \frac{\text{kg}}{\text{s}^2\cdot\text{m}}.$$

$$[\nabla p] = \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\frac{1}{\text{m}^3} = \frac{\text{kg}}{\text{s}^2\cdot\text{m}^2}.$$

the dimension matrix is

	$\mathbf{j}$	K	μ	∇p
kg	0	0	1	1
m	1	2	−1	−2
s	−1	0	−1	−2

We see that the rank is 3, and we have a dimensionless combination, which we find easily. Strictly speaking, we derive the relation for scalar quantities, and use physical insight to find the given form. It may be noted that K in general will not be a scalar, but a 2nd order tensor (This happens typically in layered rocks where the pores are mostly in one direction).

c) We assume that the rod lies along the x-axis and has a constant cross section A. The conservation law for a section of the rod is then

$$\frac{d}{dt}\int_a^b (S_i\Phi)A dx + j_i\left(b\right)A - j_i\left(a\right)A = 0,\ i = o, v.$$

We get the differential formulation in the usual way,

$$\Phi\frac{\partial S_i}{\partial t} + \frac{\partial j_i}{\partial x} = 0,\ i = o, v.$$

We eliminate $\frac{\partial p}{\partial x}$ by adding the equations for the flux:

$$-\left(k_o\left(S_o\right)\frac{K}{\mu_o} + k_v\left(S_v\right)\frac{K}{\mu_v}\right)\frac{\partial p}{\partial x} = q,$$

i.e.

$$\frac{\partial p}{\partial x} = -\frac{q}{\left(k_o\left(S_o\right)\frac{K}{\mu_o} + k_v\left(S_v\right)\frac{K}{\mu_v}\right)}.$$

By inserting this in the equation for $S \equiv S_v$, we get

$$\begin{aligned}\Phi\frac{\partial S}{\partial t} + \frac{\partial j_v}{\partial x} &= \\ &= \Phi\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}\left(k_v\left(S\right)\frac{K}{\mu_v}\frac{q}{\left(k_o\left(1-S\right)\frac{K}{\mu_o} + k_v\left(S\right)\frac{K}{\mu_v}\right)}\right) \\ &= \Phi\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}f\left(S\right) = 0.\end{aligned}$$

d) By inserting the given quantities, we see that $f\left(S\right) = qS^2$ and the equation reduces to

$$\frac{\partial S}{\partial t} + \frac{q}{\Phi}\frac{\partial S^2}{\partial x} = 0.$$

This is a regular first order hyperbolic equation with kinematic velocity

$$c(S) = \frac{2q}{\Phi}S.$$

To study this more carefully, we set up the equation for a characteristic starting point x_0, where $0 < x_0 < L$:

$$x = x_0 + \frac{2q}{\Phi}S(x_0)t = x_0 + \frac{2q}{\Phi}\left(1 - \frac{x_0}{L}\right)t.$$

We leave to the reader to plot the characteristics. If we do this correctly, we see that all the characteristics meet in the point $x_s = L,\ t_s = \frac{\Phi L}{2q}$. Furthermore, the solution $S = 1$ above the characteristic that starts in the origin, i.e. $x = \frac{2qt}{\Phi}$. For a point (x, t) below this characteristic, we have

$$x = x_0 + \frac{2q}{\Phi}\left(1 - \frac{x_0}{L}\right)t,$$

i.e.

$$x_0 = L\frac{2qt - x\Phi}{2qt - \Phi L}$$

and

$$S(x,t) = 1 - \frac{2qt - x\Phi}{2qt - \Phi L}.$$