TMA4195
Mathematical Modelling
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Solutions to exercise set 2

$\boxed{1}$ In our notation we write

$$u(x^*) = \mathrm{e}^{-10x^*} + \mathrm{e}^{-100x^*}, \quad x^* \in [0,1]$$

for the unscaled relation.

We observe that the function u follows three different behaviours within the interval $[0,1]$: At first, both terms are relevant, although the second term decreases much faster than the first one (or: the derivative of the second term dominates); then, the second term is essentially negligible, while the first one is still much larger than zero; finally, both of the terms are essentially equal to zero.

Natural scalings:

$$\begin{aligned}
&1.) \quad x^* = \frac{1}{100}x, \quad \text{then } \mathrm{e}^{-100x^*} \sim 1, \text{ when } x \sim 1;\\
&2.) \quad x^* = \frac{1}{10}x, \quad (\text{then } \mathrm{e}^{-10x^*} \sim 1 \text{ and } \mathrm{e}^{-100x^*} \ll 1, \text{ when } x \sim 1;\\
&3.) \quad x^* = 1 \cdot x, \quad (\text{then } u \sim 0, \text{ when } x \sim 1.
\end{aligned}$$

Reasonable regions for these scalings are as follows:

1. For the first scaling $x^* = \frac{1}{100}x$, we can use values $x \in [0,2]$, corresponding to $x^* \in \left[0, \frac{2}{200}\right]$. Then
$$u(x) = \mathrm{e}^{-\frac{1}{10}x_1} + \mathrm{e}^{-x_1} \approx 1 + \mathrm{e}^{-x_1}$$
or
$$u(x^*) \approx 1 + \mathrm{e}^{-100x^*}$$
2. Here we might choose values $x \in \left[\frac{2}{10}, 2\right]$, corresponding to $x^* \in \left[\frac{2}{100}, \frac{2}{10}\right]$. Then
$$u(x) = \mathrm{e}^{-x} + \mathrm{e}^{-10x} \approx \mathrm{e}^{-x}$$
or
$$u(x^*) \approx \mathrm{e}^{-10x^*}.$$
3. Here we can choose $x = x^* \in \left[\frac{2}{10}, 1\right]$, which yields
$$u(x) = \mathrm{e}^{-10x} + \mathrm{e}^{-100x} \approx 0.$$

In all cases, the boundaries between the regions should be understood to be fuzzy, and a small shift of them is perfectly fine.

$\boxed{2}$ We use the standard notation

$$(1) \qquad m^{*\prime}(t^*) = -\alpha, \qquad\qquad m^*(0) = M,$$

$$(2) \qquad x^{*\prime\prime}(t^*) = \frac{\alpha\beta}{m^*(t^*)} - \frac{g}{\left(1+\frac{x^*(t^*)}{R}\right)^2}, \qquad\qquad x^*(0) = x^{*\prime}(0) = 0.$$

Let $x^* = Rx$, $m^* = Mm$ and $t^* = Tt$ (where T is some timescale to be determined). With these scales we consequently get the relations

$$\frac{dm^*}{dt^*} = \frac{M}{T}\frac{dm}{dt}, \quad \frac{dx^*}{dt^*} = \frac{R}{T}\frac{dx}{dt}, \quad \frac{d^2x^*}{(dt^*)^2} = \frac{R}{T^2}\frac{d^2x}{(dt)^2}.$$

Inserting this in (1) and (2), and cleaning up a bit, we obtain the scaled system of equations

$$(3) \qquad m'(t) = -\frac{\alpha T}{M}, \qquad\qquad m(0) = 1,$$

$$(4) \qquad x''(t) = \underbrace{\left(\frac{T^2\alpha\beta}{RM}\right)\frac{1}{m(t)}}_{\text{Thrust}} - \underbrace{\left(\frac{gT^2}{R}\right)\frac{1}{(1+x(t))^2}}_{\text{Gravity}}, \qquad\qquad x(0) = x'(0) = 0,$$

We balance equation (4) according to when the thrust dominates over gravity (thrust$\gg$gravity); this means we set

$$\frac{T^2\alpha\beta}{RM} = 1, \implies T = \sqrt{\frac{RM}{\alpha\beta}}.$$

Thus the dimensionless problem is

$$m' = -\mu, \qquad\qquad m(0) = 1,$$

$$x'' = \frac{1}{m} - \varepsilon\frac{1}{(1+x(t))^2}, \qquad\qquad x(0) = x'(0) = 0,$$

where $\mu = \sqrt{\frac{\alpha R}{\beta M}}$ and $\varepsilon = \frac{Mg}{\alpha\beta}$. Note that ε is essentially the ration of gravitational force to rocket force meaning $\varepsilon \ll 1$ (assuming thrust dominates gravity).

$\boxed{3}$ The exact solution of the initial value problem is $y_{exact}(t) = \frac{1}{\epsilon^2}\left(1-e^{-\epsilon t}\right) - \frac{t}{\epsilon}$. Let $y_{approx}(t) = y_0(t) + \epsilon y_1(t) + \epsilon^2 y_2(t)$. We want y_{approx} to satisfy the initial data for every value of $0 < \epsilon \leq 1$. This implies that $y_0(0) = y_1(0) = y_2(0) = 0$ and $y_0'(0) = y_1'(0) = y_2'(0) = 0$. Inserting y_{approx} in the equation, we get

$$y_0'' + 1 + \epsilon\left(y_1'' + y_0'\right) + \epsilon^2\left(y_2'' + y_1'\right) = 0.$$

As this should hold for all $0 < \epsilon \leq 1$, we have

$$y_0'' + 1 = 0,$$
$$y_1'' + y_0' = 0,$$
$$y_2'' + y_1' = 0.$$

The initial values and integration of the equations gives

$$y_{approx}(t) = -\frac{1}{2}t^2 + \epsilon\frac{1}{6}t^3 - \epsilon^2\frac{1}{24}t^4.$$

The Taylor expansion of y_{exact} is given by

$$y_{exact}(t) = \frac{1}{\epsilon^2}\left(1 - \sum_{n=0}^{\infty}\frac{(-\epsilon t)^n}{n!}\right) - \frac{t}{\epsilon} = \sum_{n=2}^{\infty}\frac{\epsilon^{n-2}(-1)^{n+1}t^n}{n!}.$$

The difference between the exact and approximate solution is then

$$y_{exact} - y_{approx} = \sum_{n=5}^{\infty}\frac{\epsilon^{n-2}(-1)^{n+1}t^n}{n!} = t^2 O(\epsilon^3 t^3).$$

Thus, as long as t and ϵt are not too big, the approximation is very good. However, for large values of ϵt, the exact solution behaves like

$$y_{exact}(t) \sim -\frac{t}{\epsilon} = -\frac{1}{\epsilon^2}\epsilon t,$$

whereas the approximate solution behaves like

$$y_{approx}(t) \sim -\epsilon^2\frac{1}{24}t^4 = -\frac{1}{\epsilon^2}\frac{1}{24}\epsilon^4 t^4.$$

Thus, for large values of t, we only have a good approximation, if ϵ decreases at the same time.

$\boxed{4}$ (a) From the problem's nature we have $0 \leq v^*(t) \leq V_0$. Then, V_0 will be a scale for v^*, and moreover

$$\frac{\left|b(v^*)^2\right|}{|av^*|} \leq \frac{bV_0}{a} \ll 1.$$

We find a time scale from the simplified equation $m\frac{dv^*}{dt^*} + av^* = 0$ with solution $v^*(t^*) = A\exp\left(-\frac{a}{m}t^*\right)$, that is $T = \frac{m}{a}$. Alternatively, and this is easier, we find this scale by balancing the first and second term in equation (5) in the problem set (set $v^* = Vv$ and $t^* = Tt$):

$$m\frac{dv^*}{dt^*} \sim av^* \quad \Rightarrow \quad m\frac{V}{T}\frac{dv}{dt} \sim aVv \quad \underset{v,\dot{v}\sim 1}{\rightsquigarrow} \quad m\frac{V}{T} \sim aV \quad \Rightarrow \quad T \sim \frac{m}{a}.$$

Using this scaling, we obtain the equation in the desired form and $\varepsilon = bV_0/a \ll 1$.

(b) Plugging in $v(t) = v_0(t) + \varepsilon v_1(t) + \cdots$ into the equation, we get that

$$\begin{aligned} O\left(\varepsilon^0\right): &\quad \dot{v}_0 = -v_0, \\ O\left(\varepsilon^1\right): &\quad \dot{v}_1 = -v_1 + v_0^2. \end{aligned}$$

With the initial condition we obtain

$$\begin{aligned} v_0(t) &= \mathrm{e}^{-t}, \\ v_1(t) &= \mathrm{e}^{-t} - \mathrm{e}^{-2t}, \end{aligned}$$

or

$$v(t) = \mathrm{e}^{-t} + \varepsilon\left(\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right) + O\left(\varepsilon^2\right).$$

This is the so-called *regular perturbation.* We have seen previously that the approximative solution is not always reasonable when $t \to \infty$, and we thus need to check its long term validity.

From the theory we know that the exact solution has the form

$$v_{\mathrm{ex}}(t) = \frac{\mathrm{e}^{-t}}{1-\varepsilon\left(1-\mathrm{e}^{-t}\right)},$$

and since $0 \leq 1-\mathrm{e}^{-t} < 1$ for $t \geq 0$, we can write the solution as a convergent geometric series.

$$v_{\mathrm{ex}}(t) = \mathrm{e}^{-t}\sum_{k=0}^{\infty}\left(\varepsilon\left(1-\mathrm{e}^{-t}\right)\right)^{k}$$

The initial terms in the perturbation expansion coincide with the initial terms in the series above, and we have:

$$v_{\mathrm{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right) = \mathrm{e}^{-t}\sum_{k=2}^{\infty}\left(\varepsilon\left(1-\mathrm{e}^{-t}\right)\right)^{k} \leq \mathrm{e}^{-t}\varepsilon^{2}\sum_{m=0}^{\infty}\varepsilon^{m} = \frac{\varepsilon^{2}\mathrm{e}^{-t}}{1-\varepsilon} \leq \frac{\varepsilon^{2}}{1-\varepsilon}.$$

Thus, we have

$$\lim_{\epsilon\to 0}\left(\sup_{t>0}\left|v_{\mathrm{ex}}(t) - \left(v_0(t) + \varepsilon v_1(t)\right)\right|\right) = 0,$$

and so $v_a(t) = v_0(t) + \varepsilon v_1(t)$ is a uniform approximation to the exact solution on the domain $t > 0$.