Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4195 Mathematical Modelling Autumn 2020

Exercise set 3

$\boxed{1}$ Consider the problem

$$\ddot{x} + 2\varepsilon\dot{x} + \varepsilon x = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = 1 - \varepsilon.$$

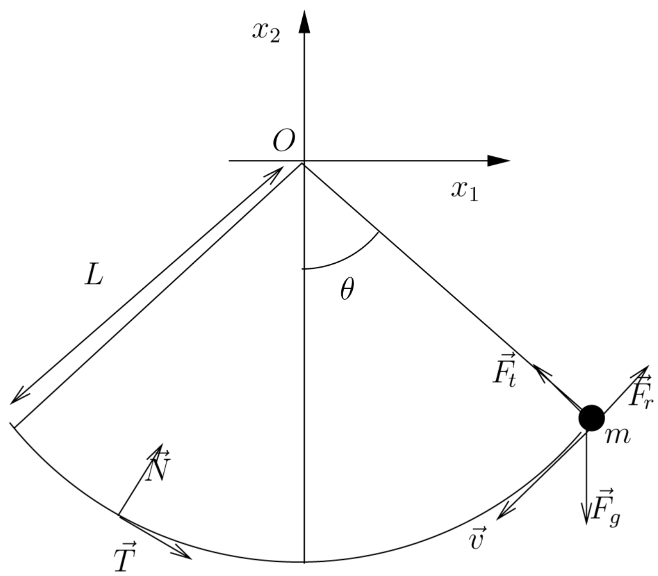
Determine x_0 and x_1 in the regular perturbation expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots .$$

$\boxed{2}$ (Problem 2, p. 23 in A. W. Bush: *Perturbation Methods for Engineers and Scientists*)
Obtain a two-term perturbation expansion for the solution of

$$\dot{y} - y - \varepsilon y^2 \mathrm{e}^{-t} = 0, \qquad y(0) = 1.$$

$\boxed{3}$ **Small angle approximation of a physical pendulum**



A pendulum of mass m and length L and point of fixation O is set in motion. The position $\vec{x}$ of the pendulum is constrained to be on the circle

$$S = \big\{\vec{x} : |\vec{x}| = L\big\}.$$

As discussed in the lecture, its motion can be described, in the absence of friction, by the equation

$$mL\frac{d^2\vartheta^*}{d(t^*)^2} = -g\sin(\vartheta^*),$$
$$\vartheta^*(0) = \Theta_0,$$
$$\frac{d\vartheta^*}{dt^*}(0) = 0.$$

After rescaling, we can rewrite this as

$$\ddot{\vartheta} = -\frac{1}{\varepsilon}\sin(\varepsilon\vartheta),$$
$$\vartheta(0) = 1,$$
$$\dot{\vartheta}(0) = 0,$$

with small parameter $0 < \varepsilon \ll 1$. Now it is possible to apply the idea of asymptotic expansions and write the solution of this equation as

$$\vartheta = \vartheta_0 + \varepsilon\vartheta_1 + \varepsilon^2\vartheta_2 + \dots$$

As we have seen in the lecture, the first terms in this expansion are

$$\begin{aligned} \vartheta_0(t) &= \cos t, \\ \vartheta_1(t) &= 0, \\ \vartheta_2(t) &= \frac{1}{192}\big(\cos t - \cos 3t\big) + \frac{1}{16}t\sin t. \end{aligned}$$

That is, we have

$$\vartheta(t) \approx \cos t + \frac{\varepsilon^2}{192}\big(\cos t - \cos 3t\big) + \frac{\varepsilon^2}{16}t\sin t$$

for small ε and t. The approximation, however, is not periodic, whereas the exact solution ϑ of the ODE is. The problem here is the term

$$\frac{\varepsilon^2}{16}t\sin t,$$

the amplitude of which grows with t. Such terms are called *secular* (in the sense of non-periodic, slowly changing with time).

In order to avoid secular terms in an asymptotic expansion, one can use the *Poincaré–Lindstedt method*. The idea here is to write

$$\vartheta(t) = \vartheta_0(\omega t) + \varepsilon\vartheta_1(\omega t) + \varepsilon^2\vartheta_2(\omega t) + \dots$$

with

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots.$$

Then one tries to find functions ϑ_0, ϑ_1, ϑ_2, ..., and values ω_1, ω_2, ..., in such a way that the secular terms are cancelled out in the approximations.

a) Find equations for ϑ_0, ϑ_1, ..., and show that

$$\vartheta_0(t) = \cos t.$$

b) Determine ϑ_1 and ϑ_2 by choosing ω_1, ω_2, ... avoiding secular terms.