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Solutions to exercise set 5

$\boxed{1}$ We consider the differential equation

$$\frac{du}{dt} = u^2(u^2 - 1).$$

whose equilibrium solutions are

$$f(u) = u^2(u^2 - 1) = 0 \quad \Rightarrow \quad u_0 = -1,\ u_1 = 0,\ u_2 = 1.$$

Let us study u_0. To show it is stable, we introduce a small perturbation $y(t)$. That is:

$$u(t) = u_0 + y(t).$$

After having plugged it in the original equation, we get

$$\frac{dy}{dt} = \underbrace{f(u_0)}_{=0} + f'(u_0)y + \frac{1}{2}f''(u_0)y^2 + \cdots .$$

Let us focus on the terms of order smaller than 1

$$\frac{dy}{dt} = f'(u_0)y$$

whose solution is

$$y(t) = Ce^{\alpha t} \quad \text{with} \quad \alpha = f'(u_0)$$

Thus, if $\alpha < 0$, the perturbation y disappears with time and the equilibrium solution u_0 is asymptotic stable. If $\alpha > 0$, the perturbation increases with time and the equilibrium solution is unstable. In our case

$$f'(u_0) = 4u_0^3 - 2u_0 = -2,$$

Therefore u_0 is a stable equilibrium solution. In the same way we can study the equilibrium solution u_2. We find that

$$f'(u_2) = 2$$

and u_2 is unstable. For u_1 we get $f'(u_1) = 0$, so we need to consider further terms in the Taylor expansion. Let us add the second one. The equation

$$\frac{dy}{dt} = \frac{1}{2}f''(u_1)y^2 = -y^2$$

has general solution

$$y(t) = \frac{1}{C+t}.$$

If we start for $t = 0$ with a small negative value for y, the constant C becomes negative and the solution blows up whenever $t \to -C$. Therefore u_1 is unstable.

$\boxed{2}$ (a) The equilibrium points are given by

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \begin{cases} u_0 = \mu \\ u_1 = \sqrt{\mu}, \quad \mu \geq 0 \\ u_2 = -\sqrt{\mu}, \quad \mu \geq 0 \end{cases}$$

We have two bifurcation points, i.e. points where the solution curves intersect. These points are given by $\mu = \pm\sqrt{\mu}$, that is the two bifurcation points are $(0,0), (1,1)$. To say something about stability, we first calculate

$$f'(u) = 3u^2 - 2\mu u - \mu$$

By plugging in the expression for the equilibrium solutions we get

$$\begin{aligned} f'(u_0) &= \mu(\mu - 1) \\ f'(u_1) &= 2\mu(1 - \sqrt{\mu}) \\ f'(u_2) &= 2\mu(1 + \sqrt{\mu}) \end{aligned}$$

We see that u_3 is always unstable, u_2 is stable for $\mu > 1$ and u_1 is stable when $0 < \mu < 1$. We see that the stability changes at the bifurcation points. See Figure 1 for the sketch of the bifurcation diagram.

(b) The equilibrium points are given by

$$\frac{du}{dt} = 0 \quad \Rightarrow \quad \begin{cases} u_0 = 0 \\ u_1 = 9/\mu, \quad \mu \neq 0 \\ u_{2,3} = 1 \pm \sqrt{\mu + 1}, \quad \mu \geq -1 \end{cases}$$

The branching diagram is showed in figure 2.
From the diagram we see that we have two bifurcation points, i.e. points where the solution curves intersect. These points are given by

$$\begin{aligned} 1 - \sqrt{\mu + 1} &= 0 \\ 1 + \sqrt{\mu + 1} &= 9/\mu \end{aligned}$$

Hence the bifurcation points are $(\mu, u) = \{(0,0), (3,3)\}$.
To say something about stability, we first calculate

$$f'(u) = (9 - \mu u)(\mu + 2u - u^2) - \mu u(\mu + 2u - u^2) + 2u(9 - \mu u)(1 - u)$$

By plugging in the expression for the equilibrium solutions we get

$$\begin{aligned} f'(u_0) &= 9\mu \\ f'(u_1) &= -9\left(\mu + \frac{18}{\mu} - \frac{81}{\mu^2}\right) \\ &= -\frac{9}{\mu^2}\left(\mu^3 + 18\mu - 81\right) \\ &= -\frac{9}{\mu^2}(\mu - 3)(\mu^2 + 3\mu + 27) \\ f'(u_2) &= 2\left(1 - \sqrt{\mu + 1}\right)\left[9 - \mu\left(1 - \sqrt{\mu + 1}\right)\right]\sqrt{\mu + 1} \\ f'(u_3) &= -2\left(1 + \sqrt{\mu + 1}\right)\left[9 - \mu\left(1 + \sqrt{\mu + 1}\right)\right]\sqrt{\mu + 1}. \end{aligned}$$

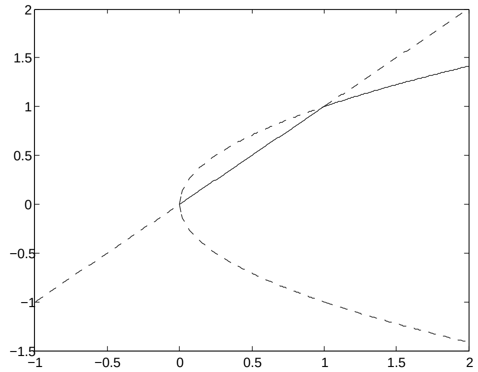

Figure 1: Bifurcation diagram 2.2 (a).

By sketching a sign line for the expression on the right hand side we find that

$$u_0 \text{ is } \begin{cases} \text{stable for } \ \mu < 0 \\ \text{unstable for } \mu > 0, \end{cases}$$

$$u_1 \text{ is } \begin{cases} \text{stable for } \ \mu > 3 \\ \text{unstable for } \mu \in \langle -\infty, 0\rangle \cup \langle 0, 3\rangle, \end{cases}$$

$$u_2 \text{ is } \begin{cases} \text{stable for } \ \mu > 0 \\ \text{unstable for } \mu \in \langle -1, 0\rangle, \end{cases}$$

$$u_3 \text{ is } \begin{cases} \text{stable for } \ \mu \in \langle -1, 3\rangle \\ \text{unstable for } \mu > 3. \end{cases}$$

We see from this that the stability changes in the bifurcation points.

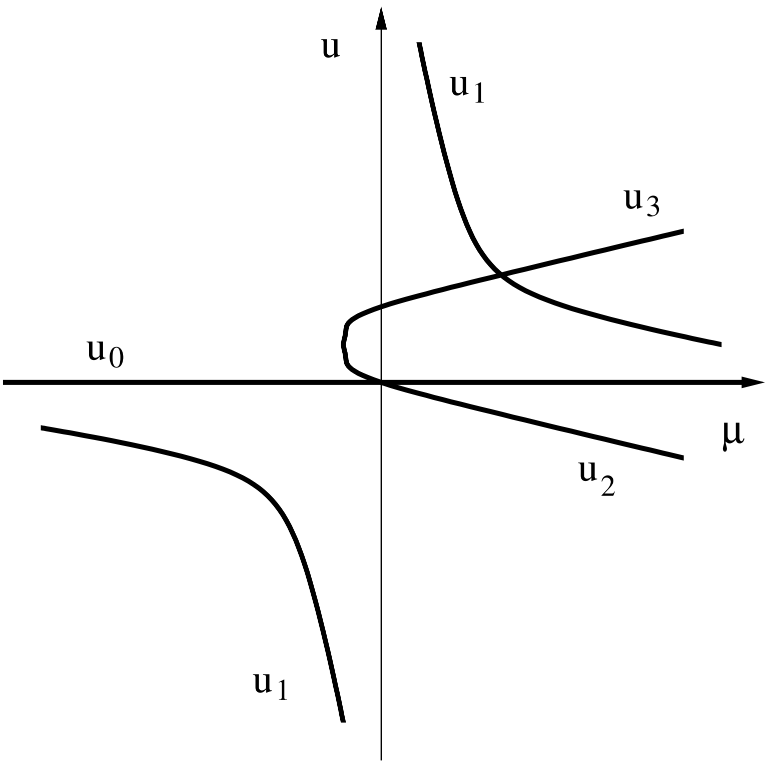

Figure 2: Branching diagram 2.2 (b).

$\boxed{3}$ We are given the equation

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + c(1-c), \quad t > 0,\ x \in \mathbb{R}. \tag{1}$$

a) We linearize the equation around $c = 0$. It is more transparent to write the equation as

$$c_t = c_{xx} + q(c).$$

The only term we need to linearize is q, since the other terms are already linear:

$$q(c) \approx q(0) + q'(0)c = 0 + 1 \cdot c = c,$$

and hence the linearized equation is

$$c_t = c_{xx} + c, \tag{2}$$

where we have set $c := c_L(x,t)$. We thus have $k = 1$.

b) We can solve the linearized equation in many ways, but following the hint, we want to transform it to the heat equation. We do this using an integrating factor. Let $\bar{c} = e^{-kt}c$ and note that

$$\frac{\partial}{\partial t}\bar{c} = e^{-kt}(c_t - kc) = e^{-kt}c_{xx} = \bar{c}_{xx}.$$

This can be solved by convolution with the fundamental solution c_F:

$$\bar{c} = \bar{c}_0 * c_F = \int_{-\infty}^{\infty} \bar{c}(y,0)c_F(x-y,t)\mathrm{d}y.$$

We then get

$$c_L(x,t) = \mathrm{e}^{kt}\bar{c}(x,t) = \mathrm{e}^{kt}\int_{-\infty}^{\infty} c_0(y)c_F(x-y,t)\mathrm{d}y,$$

where we note that $\bar{c}(y,0) = e^{-0}c(y,0) = c_0(y)$.
Inserting the given solution of the fundamental solution c_F, we get

$$c_L(x,t) = e^{kt}\int_{-\infty}^{\infty} c_0(y)\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}dy.$$

c) Using the hint, we calculate

$$|c_L - 0| \leq \mathrm{e}^{kt}\int_{-\infty}^{\infty}|c_0(y)|c_F(x-y,t)\mathrm{d}y \leq \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x)|\cdot 1 = \mathrm{e}^{kt}\max_{x\in\mathrm{R}}|c_0(x)-0|.$$

d) The equilibrium points of (1) are its constant solutions, and if $c = c_E$ is a constant solution of (1), then $(c_E)_t = (c_E)_{xx} = 0$ and $q(c_E) = c_E(c_E - 1) = 0$. The solutions/equilibrium points are therefore $c_E = 0$ and $c_E = 1$.
To study the stability of the equilibrium points c_E, we check whether solutions of the equation linearized arond c_E that start near c_E remain near for all times. To do that, let

$$c(x,t) = c_E + \tilde{c}(x,t)$$

and note that if $\tilde{c}$ is not so big, then

$$\tilde{c}_t = \tilde{c}_{xx} + q(c_E + \tilde{c}) \approx \tilde{c}_{xx} + q(c_E) + q'(c_E)\tilde{c}.$$

Note that $q(c_E) = 0$ and let $\hat{c}$ be the solution of the linearized equation

$$c_t = c_{xx} + q'(c_E)c. \tag{3}$$

This linearized equation only has the equilibrium point $\hat{c} = 0$ (since $q' \neq 0$). By definition we say that c_E is a stable(/unstable) equilibrium point of the original non-linear equation according to linear stability analysis if $\hat{c} = 0$ is a stable(/unstable) equilibrium point of the linearized equation 3.
We solve equation (3) and $c(x,0) = c_0(x)$ as in part b), this time with using the integrating factor $e^{-q'(c_e)t}$:

$$\hat{c}(x,t) = e^{q'(c_e)t}\int_{\mathbb{R}} c_0(y)c_F(x-y,t)dy.$$

Note that if $|c_0(x) - 0| = |c_0| < \delta$, then

$$|\hat{c}(x,t) - 0| \leq \max_{x\in\mathrm{R}}|c_0(x) - 0| < \delta e^{q'(c_e)t}.$$

Hence it follows that $\hat{c} = 0$ is a stable equilibrium point if $q'(c_e) \leq 0$ since then small perturbations remain small for all times. On the other hand, if $q'(c_e) > 0$, then $\hat{c} = 0$ is not stable any more since we can find small perturbations that blows up in time. Take e.g. $c_0 = \delta$ and check that

$$\hat{c}(x,t) = \delta e^{q'(c_e)t} \to \infty \qquad \text{as} \qquad t \to \infty.$$

We compute q' and find that $q'(0) = 1 > 0$ and $q'(1) = -1 < 0$. From the discussion above we can then conclude according to linear stability analysis that $c_E = 0$ is unstable while $c_E = 1$ is stable.