TMA4195
Mathematical Modelling
Autumn 2020

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Exercise set 6

1 (Exercise 9.2.7 in Krogstad)
Consider the following simple model for the moose population (P) in Trøndelag:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(\frac{P}{m} - 1\right), \quad 0 < m < M, \quad k > 0$$

(a) What properties does the model try to explain, and what are the equilibrium populations? Determine, using linear stability analysis, whether they are stable or unstable. Draw a simple sketch showing the solutions.

A simplified model, also including the hunter population (J) has, after a proper scaling, the form

$$\begin{aligned} \frac{dP}{dt} &= P(1-P) - J \\ \frac{dJ}{dt} &= -\frac{J}{2} + JP \end{aligned} \tag{1}$$

(b) Determine the equilibrium points for (1) and of what kind they are (or appear to be).

(c) Show that the solution of

$$h\left(P, J\right) = J - 3P(1-P)/2 = 0$$

defines a possible phase-space path for the model in (1).
Hint: Show that ∇h along the path is always orthogonal to the direction of the motion defined by $\left(\frac{dP}{dt}, \frac{dJ}{dt}\right)$. Why is this smart to consider?

(d) The point $(P_c, J_c) = \left(\frac{1}{2}, \frac{1}{4}\right)$ could be a center for (1). Shift the origin by introducing

$$\begin{aligned} x &= P - 1/2, \\ y &= J - 1/4, \end{aligned}$$

and show that the system has paths which are symmetric about the y-axis and therefore that the point is really a center.

(e) Summarize the qualitative properties of the model in (1).

2 (Exercise 9.2.8 p. 20 in Krogstad) In a population consisting of M_0 individuals, an influenza infection is transferred by sick persons ($S^*(t)$ individuals) meeting susceptible persons ($M_0 - S^*(t)$ individuals). The number of people infected per time unit is proportional to the probability that sick persons meet susceptible persons. The proportionality constant, r is called the *infection rate* (per individual). After a while, the sick individuals recover.

(a) The following dynamic model has been proposed for the number of sick persons in the population if one assumes that none are immune:

$$\frac{dS^*}{dt^*} = rS^*(M_0 - S^*) - \alpha S^*.$$

Consider a simplified equation in the early stages of the epidemic and find suitable scales for the variables.

A more realistic model takes into account that those who have been cured will be immune, at least for some time after the illness. The following (scaled) mathematical model has been proposed for the number of sick, $S(t)$, and immune, $I(t)$, persons:

$$\frac{dS}{dt} = S(1 - I - S) - \lambda S,$$
$$\frac{dI}{dt} = \lambda S - \mu I.$$

(b) State the region in the (S, I)-plane for physically acceptable solutions, and the meaning of the parameters μ and λ. This dynamical system has an obvious stationary point. When is this point stable?
It is not necessary to study limit cases.

(c) Determine for which other values of μ and λ the system has another physically acceptable stationary point (S_0, I_0). Show that this point moves on a line segment if λ is kept fixed while μ varies.

(d) Linearize the system around (S_0, I_0) by introducing $S(t) = S_0 + x(t)$, $I(t) = I_0 + y(t)$ and show that the matrix $\mathbf{A}$ in the linearized system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} = [x, y]^T$, has the form

$$\mathbf{A} = \begin{bmatrix} -S_0 & -S_0 \\ \lambda & -\mu \end{bmatrix}.$$

Determine whether (S_0, I_0) is stable or unstable.

3 (Problem 1 from the exam 2010.)
Let $0 < \alpha < r$, $N_0 > 0$, and consider the following initial value problem,

$$\begin{cases} \dfrac{dN^*(t^*)}{dt^*} = rN^*(t^*)\Big(1 - \dfrac{N^*(t^*)}{K}\Big) - \alpha N^*(t^*), \quad t^* > 0, \\ N^*(0) = N_0. \end{cases} \tag{2}$$

(a) What could problem (2) model?
Find reasonable scales for this problem when $N_0 \gg K$ and $\alpha < r$.

Let $\varepsilon, \alpha, \beta > 0$, $\alpha, \beta < 1$, and consider the following scaled system of equations,

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x) - \alpha xy, \\ \varepsilon \dfrac{dy}{dt} = y(1 - y) - \beta xy. \end{cases} \tag{3}$$

(b) What could system (3) model?
Find the equilibrium points and determine their stability when $\beta = 0$.