TMA4195
Mathematical Modelling
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Solutions to exercise set 6

1 **(a)** We consider the following model for the population (P) of moose

$$\frac{dP}{dt} = f(P), \tag{1}$$

$$f(P) = kP\left(1 - \frac{P}{M}\right)\left(\frac{P}{m} - 1\right) \tag{2}$$

for $0 < m < M$. The equilibrium points are given by

$$f(P) = 0 \quad \Rightarrow \quad P_1 = 0,\ P_2 = m,\ P_3 = M. \tag{3}$$

Their stability properties can be determined using the derivative

$$f'(P) = k\left(1 - \frac{P}{M}\right)\left(\frac{P}{m} - 1\right) - \frac{k}{M}P\left(\frac{P}{m} - 1\right) + \frac{k}{m}P\left(1 - \frac{P}{M}\right)$$

and we obtain

$$\begin{aligned}
P_1 = 0 \quad &: \quad f'(P_1) = -k < 0 && \Rightarrow \quad \text{stable} \\
P_2 = m \quad &: \quad f'(P_2) = k\underbrace{\left(1 - \frac{m}{M}\right)}_{>0} > 0 && \Rightarrow \quad \text{unstable} \\
P_3 = M \quad &: \quad f'(P_3) = -k\underbrace{\left(\frac{M}{m} - 1\right)}_{>0} < 0 && \Rightarrow \quad \text{stable}
\end{aligned}$$

(It is even simpler to obtain the same from a sketch of the 3rd order polynomial $f(P)$.)

The situation is illustrated in figure 1. From the plot we see the main features of the model:

- m is the minimum population of moose able to survive: a smaller amount would die out.
- M is the biggest viable amount of moose: the population cannot be bigger than M for a long time, reflecting, for example, limited available resources.

(b) Let us consider a model where hunters (J) are also present:

$$\frac{dP}{dt} = P(1-P) - J = F(P,J),$$
$$\frac{dJ}{dt} = -\frac{J}{2} + JP = G(P,J).$$

The singular (or equilibrium) points are given by

$$\begin{cases} F(P,J) = P(1-P) - J = 0 \\ G(P,J) = J(P - 1/2) = 0 \end{cases} \Rightarrow \begin{cases} J = P(1-P), \\ P(1-P)(P-1/2) = 0, \end{cases}$$

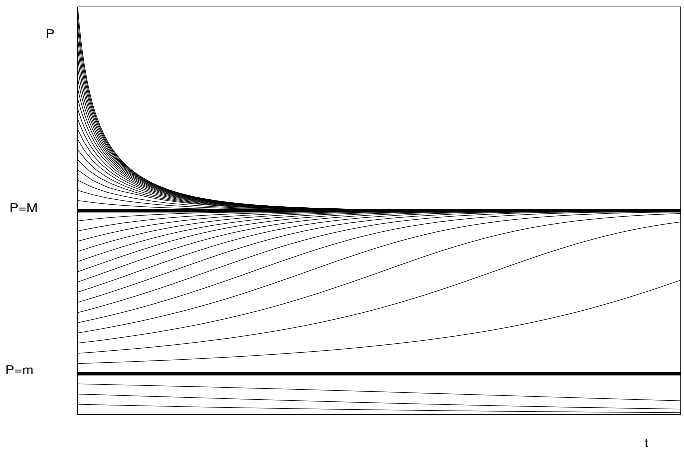

Figure 1: Sketch of solutions.

that is, $(P_0, J_0) = \{(0,0), (1/2, 1/4), (1,0)\}$. We shall use linear analysis to determine their stability and start by computing the matrix A in the Taylor expansion of the right hand side,

$$\begin{pmatrix} F \\ G \end{pmatrix}_{(P_0,J_0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + (A)_{(P_0,J_0)} \begin{pmatrix} P-P_0 \\ J-J_0 \end{pmatrix} + \text{ higher order terms.}$$

Thus,

$$A = \begin{pmatrix} \partial F/\partial P & \partial F/\partial J \\ \partial G/\partial P & \partial G/\partial J \end{pmatrix} = \begin{pmatrix} 1-2P & -1 \\ J & -\frac{1}{2}+P \end{pmatrix}.$$

For $(0,0)$ the matrix is

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -\frac{1}{2} \end{pmatrix},$$

which implies that $\lambda_1 = 1,\ \lambda_2 = -\frac{1}{2}$ and $(0,0)$ is a *saddle-point.*
For $(1/2, 1/4)$ we obtain

$$A = \begin{pmatrix} 0 & -1 \\ \frac{1}{4} & 0 \end{pmatrix} \Rightarrow \lambda_1 = -\frac{1}{2}i,\ \lambda_2 = \frac{1}{2}i,$$

showing that $(1/2, 1/4)$ is a *center* by the linear theory (but remember that this is not enough to deduce something about the stability of the non-linear system). For $(1,0)$ we obtain that

$$A = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow \lambda_1 = -1,\ \lambda_2 = \frac{1}{2}.$$

Also $(1,0)$ is a *saddle-point.*

(c) We are now going to show that the solution of

$$h(P,J) = J - 3P(1-P)/2 = 0 \tag{4}$$

defines a trajectory for the dynamical system. This will be the case if ∇h and $\left(\frac{dP}{dt}, \frac{dJ}{dt}\right)$ are orthogonal. (The argument is as follows: The direction of the motion is defined by the velocity vector $\left(\frac{dP}{dt}, \frac{dJ}{dt}\right)$ in all points. Moreover, the solution of Eqn. 4 is a level curve for the surface $z = h\,(P, Q)$, and the gradient ∇h at a point on the curve is always orthogonal to the level curve. Thus, if ∇h and $\left(\frac{dP}{dt}, \frac{dJ}{dt}\right)$ are orthogonal, a solution starting on the level curve can never leave it.) Now, ∇h computes as

$$\nabla h = \begin{pmatrix} 3P - \frac{3}{2} \\ 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}
\nabla h \cdot \begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} &= (3P - \frac{3}{2}) \cdot (P(1-P) - J) + 1 \cdot J(-\frac{1}{2} + P) \\
&= -\frac{3}{2}\,(1 - 2P)\left(-\frac{1}{2}P(1-P)\right) + \frac{3P(1-P)}{2}(-\frac{1}{2} + P) \\
&= 0.
\end{aligned}$$

The solution given by $h = 0$ is drawn in figure 2 as a thick black curve. Since

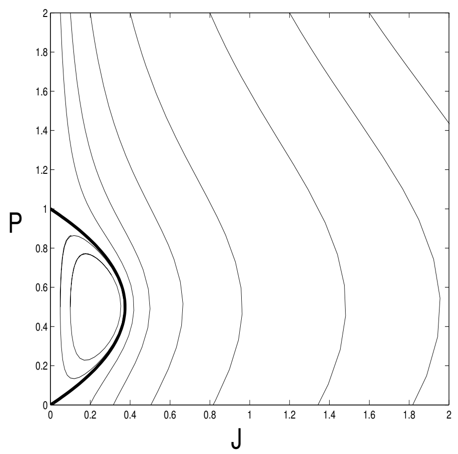

Figure 2: Plot of solutions.

the solutions cannot cross (remember that our dynamical system returns a unique solution for every starting point), every initial condition that is located inside the area delimited by $h\,(P, J) = 0$ will give rise to a solution that stays inside the same area for all times. From the equation describing the moose's population we observe that

$$\frac{dP}{dt} < 0 \text{ for } J > P(1-P).$$

and, in particular,

$$\frac{dP}{dt} < 0 \text{ for } J > \frac{3}{2}P(1-P).$$

Therefore, all the solutions outside the region given by $h = 0$ will die out. The figure 2 is made by solving the dynamical system numerically.

(d) From the plot we observe that the solutions inside the area delimited by $h = 0$ appear to be periodic, but this is not sufficient as a mathematical proof. Let us therefore define new coordinates (x, y) as

$$\begin{cases} x = P - \frac{1}{2} \\ y = J - \frac{1}{4}. \end{cases}$$

(so that $(\frac{1}{2}, \frac{1}{4})$ lies in the new origin). With these coordinates, our system becomes

$$\frac{dx}{dt} = -x^2 - y$$
$$\frac{dy}{dt} = x\left(y + \frac{1}{4}\right).$$

There are several ways to argue from here. For a certain point (x, y), $x \neq 0$, the derivative

$$\frac{dx}{dy} = \frac{-x^2 - y}{x\left(y + \frac{1}{4}\right)}$$

satisfies

$$\frac{dx}{dy}(x, y) = -\frac{dx}{dy}(-x, y).$$

This means that the trajectories through the points (x, y) and $(-x, y)$ are mirror images of each other w.r.t. the y-axes. Such a symmetry would not be possible in the case of spirals (what happens when $x \to 0$?). Thus, the solutions have to be periodic. Of course, the symmetry shows up clearly in figure 2, where the solutions are symmetric with respect to the line $P = 1/2$.

(e) Conclusions from our model:

- The hunters reduce the amount of animals.
- $\begin{cases} P > 1/2 : \text{It is actually interesting to hunt, } J \text{ increases.} \\ P < 1/2 : \text{It is not so interesting to hunt, } J \text{ decreases.} \end{cases}$

2 (a) A natural scale for S^* is given by $S^* = M_0 S$ as $S^* \leq M_0$. Let $t^* = Tt$, and assume that $S^* \ll M_0$ (early stage). Then we balance $\frac{\mathrm{d}S^*}{\mathrm{d}t^*} \sim rS^*(M_0 - S^*)$ and get $T = \frac{1}{rM_0}$. The scaled equation is then

$$\frac{\mathrm{d}S}{\mathrm{d}t} = S(1 - S) - \lambda S,$$

where $\lambda = \frac{\alpha}{rM_0}$. There are two stationary solutions, i.e. solutions S such that $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$, given by $S_1 = 0$ and $S_2 = 1 - \lambda$. These make sense physically. If there are no sick persons, it should continue this way. And if the infection rate is very high the other stationary solution should be close to 1.

(b) The physically acceptable region is $0 \leq S, I, S + I \leq 1$. The parameter λ compares the rate people recover to the rate people get sick. In similar fashion, the parameter μ tells us something about the ratio of the rate people get sick and the rate people lose immunity. The obvious stationary point is $(0, 0)$. The

Jacobi matrix at $(0,0)$ is given by $\begin{pmatrix} 1-\lambda & 0 \\ \lambda & -\mu \end{pmatrix}$, with eigenvalues $1-\lambda, -\mu$. Both λ and μ are nonnegative, and this gives that both eigenvalues are negative for $\lambda > 1$. So $(0,0)$ is stable if $\lambda > 1$ and unstable else.

(c) We solve

$$\begin{aligned} \lambda S_0 - \mu I_0 &= 0, \\ S_0(1 - I_0 - S_0) - \lambda S_0 &= 0. \end{aligned}$$

The above gives $S_0 = \frac{\mu}{\lambda} I_0$, and as we are looking for a solution other than $(0,0)$, we can assume that $S_0 \neq 0$, and we can divide by S_0. This gives $1 - I_0 - S_0 - \lambda = 0$, or

$$S_0 + I_0 = 1 - \lambda. \tag{5}$$

Inserting $S_0 = \frac{\mu}{\lambda} I_0$ in the system yields

$$\begin{aligned} I_0 &= \frac{\lambda}{\lambda + \mu}(1 - \lambda), \\ S_0 &= \frac{\mu}{\lambda + \mu}(1 - \lambda). \end{aligned}$$

Acceptable values for λ and μ are thus $0 < \lambda < 1$ and $0 < \mu < \infty$. For constant λ the line is given by (5).

(d) The linearisation about (S_0, I_0) is found by defining the function $F(S, I) = (f(S, I), g(S, I))$ to be the right hand side in the system of ODEs and finding the Jacobian $J_{(S_0,I_0)}F$ at the point (S_0, I_0). The Jacobian matrix is

$$\begin{aligned} J_{(S_0,I_0)}F &= \begin{pmatrix} \frac{\partial f}{\partial S}(S_0, I_0) & \frac{\partial f}{\partial I}(S_0, I_0) \\ \frac{\partial g}{\partial S}(S_0, I_0) & \frac{\partial g}{\partial I}(S_0, I_0) \end{pmatrix} \\ &= \begin{pmatrix} 1 - \lambda - I_0 - 2S_0 & -S_0 \\ \lambda & -\mu \end{pmatrix} \end{aligned}$$

We know that $1 - \lambda = I_0 + S_0$.

$$J_{(S_0,I_0)}F = \begin{pmatrix} -S_0 & -S_0 \\ \lambda & -\mu \end{pmatrix}. \tag{6}$$

We compute the eigenvalues α_i as usual,

$$\alpha_\pm = \frac{-(\mu + S_0) \pm \sqrt{(\mu + S_0)^2 - 4(\mu + \lambda)S_0}}{2}.$$

The real part of α is negative and the equilibrium is stable.

3 (a) When $\alpha = 0$ equation (5) is the logistic equation for population growth with growth rate r and carrying capacity K. The α-term is a harvest or death term.

<u>Scales</u>:

Since $N_0 > K$, we see that $\dfrac{dN^*}{dt^*} < 0$ for $t \ll 1$, and hence $\max N^* = N_0$.

Take $N^* = N_0 N$, $t^* = Tt$, scale (1), and balance terms:

$$\frac{dN^*}{dt^*} = \frac{N_0}{T}\frac{dN}{dt} \overset{(1)}{=} rN_0N\left(1 - \frac{N_0}{K}N\right) - \alpha N_0 N$$
$$\Rightarrow T = \frac{1}{r\left(1 - \dfrac{N_0}{K}\right) - \alpha} \sim \frac{K}{rN_0}$$

since $\alpha < r$ and $\frac{N_0}{K} \gg 1$.
Scales: $N^* = N_0 N$, $t^* = \frac{K}{rN_0}t$.

(b) System (6) models e.g. growth of 2 populations sharing limited resource. The death-rates (α, β-terms) depend on the size of the competing population. When $\alpha, \beta = 0$, the populations experience logistic growth.
Equilibrium points, when $\beta = 0$

$$0 = \frac{dx}{dt} = f_1(x,y) = x(1-x) - \alpha xy \quad \Rightarrow x = 0 \text{ or } x = 1 - \alpha y$$
$$0 = \varepsilon\frac{dx}{dt} = f_2(x,y) = y(1-y) \quad \Rightarrow y = 0 \text{ or } y = 1$$

Solutions = equilibrium points:

$$(x,y) = \{(0,0), (0,1), (1,0), (1-\alpha, 1)\}.$$

Stability:

$$\text{Jacobian} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 - 2x - \alpha y & -\alpha \\ 0 & 1 - 2y \end{bmatrix},$$

and the eigenvalues are $\lambda_1 = 1 - 2x - \alpha y, \lambda_2 = 1 - 2y$. Hence

	$(0,0)$	$(0,1)$	$(1,0)$	$(1-\alpha,1)$
λ_1	1	$1-\alpha(>0)$	-1	$-(1-\alpha)(<0)$
λ_2	1	-1	1	-1
	unstable	unstable	unstable	stable