75048 Mathematical modeling 11 January 1999

Suggested solution

Problem 1

a Any physically meaningful relation $\Phi(R_1, \ldots, R_n) = 0$ (with $R_j \neq 0$) is equivalent to a relation of the form $\Psi(\pi_1, \ldots, \pi_{n-r}) = 0$ involving (a maximal set of) independent dimensionless combinations of the given variables.

b Relevant quantities and their units for waves in deep water:

Quantity	V	L	g	ρ
Units	$\mathrm{m\,s^{-1}}$	m	$\mathrm{m\,s^{-2}}$	$\mathrm{kg\,m^{-3}}$

The only dimensionless combination of these is $V^2/(Lg)$ (and powers thereof). By Buckingham's pi theorem the only physically meaningful relation between these is can be written $V^2/(Lg) = \text{constant}$, which can also be written as

$$V = C\sqrt{Lg}.$$

For waves dominated by surface tension the table looks like

Quantity	V	L	σ	ρ
Units	$\mathrm{m\,s^{-1}}$	m	$\mathrm{kg\,s^{-2}}$	$\mathrm{kg\,m^{-3}}$

whose single dimensionless combination is $\rho V^2 L/\sigma$, which leads to

$$V = C\frac{\sigma}{\rho L}.$$

c The critical length L_0 must be a function of g, ρ, and σ, whose single dimensionless combination is $\rho g L_0^2/\sigma$. So we expect the critical length to be given by a particular value of this expression. Having no further information, we might guess this value to be close to 1, leading to

$$L_0 \approx \sqrt{\frac{\sigma}{\rho g}} \approx 2.6\,\mathrm{mm}.$$

Problem 2

A noticeable proportion of the heat input must penetrate to a depth of $L \approx 1\,\mathrm{cm}$ in $t \approx 30\,\mathrm{s}$ for the measurement to satisfy the given requirements. But how to estimate this depth? Thermal properties for water, fat, and protein have been given, and we must expect the properties for meat or fish to be some sort of average of these: Neither larger than the largest of the individual values, nor smaller than the smallest. The only dimensionless combination of the quantities L, t, ρ, c, k is $kt/(\rho c L^2)$ (or powers thereof), leading to the estimate

$$L \approx \sqrt{\frac{kt}{\rho c}}$$

in which we optimistically insert the largest available value for k (that for water) and the smallest one for c (protein) together with $t = 30\,\mathrm{s}$, or

$$L \approx \sqrt{\frac{0.56 \cdot 30}{1000 \cdot 1300}}\,\mathrm{m} \approx 3.6\,\mathrm{mm}$$

which is quite a bit too small, particularly because the heat must not only penetrate three times deeper, but the influence of the thermal properties at that depth must make it back to the surface where it can be measured. Thus we conclude that this device is likely to only at best yield properties of the upper millimeter or two of the sample, and so fails the requirements.

We should note that in the above discussion we replaced a constant arising from dimensional analysis by 1 without further comment. However, in the case of the heat equation and its fundamental solution (in one space dimension)

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2}, \qquad \Phi(x,t) = \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{x^2}{4t}\Big)$$

this practice can be defended – or maybe we should use twice the value, as the exponent equals -1 just when $x = 2\sqrt{t}$ – but this does not change the conclusion very much.

One can also arrive more directly at these conclusions by recognizing that the non-scaled heat equation can be written in the form

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c} \Delta T$$

where the diffusivity (the fraction on the right hand side) has units $\mathrm{m\,s^{-1}}$, scaling the equation correspondingly, and using the above-mentioned properties of the solutions of the heat equation.

Problem 3

Writing u' for the derivative of u with respect to τ, note that $\ddot{u} = u''/(1 + \varepsilon c_1 + \mathcal{O}(\varepsilon^2))^2 = u'' \cdot (1 - 2\varepsilon c_1 + \mathcal{O}(\varepsilon^2))$. Put now $u = u_0 + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$ in Duffing's equation, and rearrange to get

$$(1 - 2\varepsilon c_1)(u_0'' + \varepsilon u_1'') + (u_0 + \varepsilon u_1 + \varepsilon u_0^3) = \mathcal{O}(\varepsilon^2)$$

which, after expanding and collecting like powers of ε, yields the two equations

$$u_0'' + u_0 = 0, \quad u_1'' + u_1 - 2c_1 u_0'' + u_0^3 = 0$$

The given boundary conditions imply

$$u_0(0) = a, \quad u_0'(0) = u_1(0) = u_1'(0) = 0.$$

Thus we arrive at

$$u_0(\tau) = a\cos\tau.$$

Furthermore, the u_1 equation now becomes

$$u_1'' + u_1 + \Big(\frac{3}{4}a^3 + 2ac_1\Big)\cos\tau + \frac{1}{3}a^3\cos 3\tau = 0$$

which, with the above initial values, has the solution

$$u_1(\tau) = \frac{1}{24}a^3(\cos\tau - \cos 3\tau) - \underbrace{\Big(ac_1 + \frac{3}{8}a^3\Big)\tau\sin\tau}_{\text{secular term}}.$$

Clearly, the secular term vanishes when $c_0 = -3a^2/8$, which leads to the estimated period:

$$T = 2\pi\Big(1 - \frac{3}{8}a^2\varepsilon + \mathcal{O}(\varepsilon^2)\Big).$$

Problem 4

Try first simply setting $\varepsilon = 0$, which immediately yields the "outer" solution

$$u_0(x) = \frac{1}{2 - x^2}$$

which does satisfy the boundary condition at $x = 0$, but not the one at $x = 1$. So we look for a boundary layer at $x = 1$. Try $x = 1 - \delta X$. Putting $u(x) = U(X) = U\big((1 - x)/\delta\big)$ in the equation, we arrive at

$$\frac{\varepsilon}{\delta^2}U'' - \big(1 + 2\delta X - (\delta X)^2\big)U + 1 = 0, \quad U(0) = 0$$

(The initial condition for U comes from the boundary condition for u at $x = 1$.) We should put $\delta = \sqrt{\varepsilon}$ to allow the first term to balance the last term in the equation. Then replacing the δ by 0, we get the equation for the lowest order approximation of the inner function: $U_0'' - U_0 + 1 = 0$. This has the general solution $U_0 = 1 + Ae^X + Be^{-X}$. The initial condition $U_0(0) = 0$ tells us that $1 + A + B = 0$. Now we match this to the outer solution:

$$\lim_{X\to\infty} U_0(X) = \lim_{x\to 1} u_0(x) = 1$$

which (fortunately!) can be satisfied by setting $A = 0$. Thus $B = -1$, and we obtain the lowest order approximation by adding together the outer and inner solution, then subtracting their common part (which is 1):

$$u \approx \frac{1}{2 - x^2} - \exp\frac{x - 1}{\sqrt{\varepsilon}}.$$

Problem 5

a Energy balance in one resistor:

$$C\frac{dT_j^*}{dt^*} = R^*(T_j^*)(I_j^*)^2 - A(T_j^* - T_a) - B(T_j^* - T_{3-j}^*) \tag{1}$$

The lefthand side is the rate of increase of thermal energy in the resistor. This balances the rate of added energy from the current in the resistor (the first term on the righthand side) and lost heat to the surroundings (the second term, assuming the surroundings have a temperature T_a) and to the other resistor (final term).

To this equation we must add the two equations

$$R^*(T_1^*)I_1^* = R^*(T_2^*)I_2^*, \quad I_1^* + I_2^* = I. \tag{2}$$

(The voltage drop across the two resistors must be the same, and the sum of the two currents is I.)

We pick scalings given by

$$I_j^* = II_j, \quad T_j^* = T_a + \Theta T_j, \quad t^* = \frac{C}{A}t.$$

It seems reasonable to scale currents so, since the non-dimensional current I_j then takes values between 0 and 1. Similarly we added the constant T_a in the expression for T_j^*, since non-dimensional temperature $T_j = 0$ then corresponds to the surrounding temperature. (We cannot give a value for the scaling factor Θ for temperature here, since not enough information has been given. But this value should be chosen so that $R(T)$ becomes a well scaled function.) There were at least two likely choices for the time scale. We chose the time scale related to cooling due to heat transfer to the surroundings, but we might equally reasonably have chosen the time scale C/B related to the heat transfer between the resistors.

Substituting all our scalings into (1) and dividing by $A\Theta$ yields

$$\frac{dT_j}{dt} = \underbrace{\frac{I^2}{A\Theta}R^*(T_j^*)}_{R(T_j)} I_j^2 - T_j - \beta(T_j - T_{3-j}), \qquad \beta = \frac{B}{A}.$$

Here, the proper scale of resistance follows from the equation as shown. The same scalings substituted in (2) produces the other two equations of the given scaled model.

b Solving the algebraic equations ($R(T_1)I_1 = R(T_2)I_2$ and $I_1 + I_2 = 1$) for I_1 and I_2 we get

$$I_j = \frac{R(T_{3-j})}{R_1 + R_2} \qquad (j = 1, 2)$$

which, when substituted into the differential equations, yields the system

$$\dot{T}_j = \frac{R(T_j)R(T_{3-j})^2}{\big(R(T_1) + R(T_2)\big)^2} - T_j - \beta(T_j - T_{3-j}) \qquad (j = 1, 2).$$

The linearization of this at a presumed symmetric equilibrium point will be given by the Jacobian of the righthand side:

$$M = \begin{bmatrix} -(\beta+1) & \beta + R'/4 \\ \beta + R'/4 & -(\beta+1) \end{bmatrix}$$

where $R' = R'(T_1) = R'(T_2)$. The characteristic equation for this matrix is $(\beta + 1 + \lambda)^2 - (\beta + R'/4)2$, which has roots

$$\lambda = -\beta - 1 \pm (\beta + R'/4) \qquad (\lambda = -1 + R'/4, \quad \lambda = -1 - 2\beta - R'/4).$$

The equilibrium is stable if and only if both these eigenvalues are negative, in other words $-1 + R'/4 < 0$ and $-1 - 2\beta - R'/4 < 0$. These two equations are equivalent to $-4(1+2\beta) < R' < 4$.

Problem 6

a The speed of individual cars is assumed to be a decreasing function of the traffic density u^*. If cars on a nearly empty road manage a top speed of V and traffic comes to a complete stop when the density is $u_{\max}$, then this model simply draws a straight line, assuming that the speed of the individual car is $V \cdot (1 - u^*/u_{\max})$. The corresponding flux of cars is obtained by multiplying this by the density u^*, leading to the flux function $f^*(u^*) = V \cdot u^* \cdot (1 - u^*/u_{\max})$. The corresponding conservation law is (on differential form)

$$\frac{\partial u^*}{\partial t^*} + \partial f^*(u^*) \partial x^* = 0.$$

Clearly, a suitable scale for u^* is $u_{\max}$, so we put $u^* = u_{\max} u$ where u is the dimensionless density. Introduce a time scale T and a length scale X via $t^* = Tt$ and $x^* = Xx$. Introduce these scales in the above equation:

$$\frac{u_{\max}}{T} \frac{\partial u}{\partial t} + \frac{1}{X} \frac{\partial}{\partial x} \big(V \cdot u_{\max} u (1 - u) \big) = 0.$$

Multiply this by $T/u_{\max}$ in order to arrive at the desired

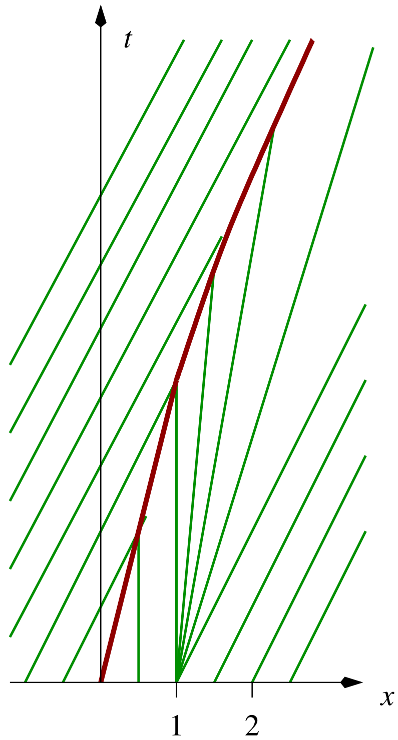
$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad f(u) = u - u^2$$

provided the scales have been chosen so that $V = XT$.

b The Rankine-Hugoniot condition states that the shock speed σ of a shock with left and right values u_l and u_r satisfies the relation

$$\sigma = \frac{f(u_r) - f(u_l)}{u_r - u_l}.$$

Remember that characteristics have speed $f'(u) = 1 - 2u$. Particular speeds of interest for this example are $f'(\frac{1}{2}) = 0$ and $f'(\frac{1}{4}) = \frac{1}{2}$. We see that a shock arises at $x = 0$ with speed $\big(f(\frac{1}{2}) - f(\frac{1}{4})\big)/(\frac{1}{2} - \frac{1}{4}) = \frac{1}{4}$. A rarefaction wave centered at $x = 1$ covers speeds from 0 to $\frac{1}{2}$, so the lefthand edge of that wave is stationary, and is hit by the shock wave at time $t = 4$. After that time the shock will slow down and weaken, as its left state $u_l = \frac{1}{4}$ (a constant) while its right state u_r decreases from $u_r = \frac{1}{2}$ towards $\frac{1}{4}$. (The picture on the right is not computed, but drawn with a drawing program.)



c From left to right in the picture below are characteristics and shocks for the three cases $0 < u_0 < \frac{1}{2} - \frac{1}{4}\sqrt{2}$, $\frac{1}{2} - \frac{1}{4}\sqrt{2} < u_0 < \frac{1}{2}$, and $\frac{1}{2} < u_0 < 1$. The constant state in the left picture solves $f(u) = 2f(u_0)$ with $u_0 < u < \frac{1}{2}$. The constant state in the middle picture is $u = \frac{1}{2} + \frac{1}{4}\sqrt{2}$, which solves $f(u) = \frac{1}{4}$. And that in the righthand picture solves $2f(u) = f(u_0)$, $u_0 < u < 1$.