Problem 1:
The logistic equation models population growth:
\[
\frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right),
\]
where \(N\) is the size of the population, \(r\) the growth rate, and \(K\) the carrying capacity.

The Lotka–Volterra system models predator (\(y\)) and prey (\(x\)) populations and how they interact.

Linear stability analysis:
\[
f = \begin{bmatrix} x(1-y) \\ ay(-1+x) \end{bmatrix} \implies Df = \begin{bmatrix} 1-y & -x \\ ay & a(-1+x) \end{bmatrix}.
\]
We observe that \(Df(0,0)\) has eigenvalues \(\lambda_1 = 1, \lambda_2 = -\alpha\), so since \(\max_{i=1,2} \Re \lambda_i > 0\), \((0,0)\) is an unstable equilibrium point. The matrix \(Df(1,1)\) has eigenvalues \(\lambda_i = \pm i\sqrt{\alpha}\), so \(\max_{i=1,2} \Re \lambda_i = 0\) and we get no conclusion from linear stability analysis.

Problem 2:
The dimension matrix \(A\) is

\[
\begin{array}{cccc}
1 & -3 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 \\
0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

Since \(\text{rank } A = 3\), we get \(5 - 3 = 2\) dimensionless combinations. Trial and error quickly leads to
\[
\pi_1 = N, \quad \text{and} \quad \pi_2 = \frac{\rho}{\sigma} U^2 r.
\]
If there is a relation \(\Phi(N, r, \rho, U, \sigma) = 0\), Buckingham’s \(\pi\)-theorem tells us that there is an equivalent dimensionally consistent relation \(\Psi(\pi_1, \pi_2) = 0\). Solving for \(N\), we find that
\[
N = \tilde{\Psi}(\pi_2) = \tilde{\Psi} \left(\frac{\rho r U^2}{\sigma}\right),
\]
for some unknown function \(\tilde{\Psi}\).
Problem 3:

a) Simply insert the perturbation expansions into the initial value problem and equate terms of equal order in $\varepsilon$. This leads to the following four initial value problems:

\[
\ddot{x}_0 = 0, \quad x_0(0) = 0, \quad \dot{x}_0(0) = 1, \\
\ddot{x}_1 = -\dot{x}_0 \sqrt{\dot{x}_0^2 + \dot{x}_1^2}, \quad x_1(0) = \dot{x}_1(0) = 0, \\
\ddot{z}_0 = -1, \quad z_0(0) = 0, \quad \dot{z}_0(0) = 1, \\
\ddot{z}_1 = -\dot{z}_0 \sqrt{\dot{z}_0^2 + \dot{z}_1^2}, \quad z_1(0) = \dot{z}_1(0) = 0.
\]

We solve for $x_0$ and $z_0$ and get

\[
x_0 = t, \quad z_0 = -\frac{1}{2} t^2 + t.
\]

b) Since $\vec{v} = (\dot{x}, \dot{z})$, Newton's second law gives us

\[
m \begin{bmatrix} \ddot{x} \\
\ddot{z}
\end{bmatrix} = \vec{F}_g + \vec{F}_r = -mg \begin{bmatrix} 0 \\
1
\end{bmatrix} - c \begin{bmatrix} \dot{x} \\
\dot{z}
\end{bmatrix} \sqrt{\dot{x}^2 + \dot{z}^2},
\]

with initial conditions $(\dot{x}(0), \dot{z}(0)) = (0, 0), (\ddot{x}(0), \ddot{z}(0)) = (U_0, U_0)$.

We use the scaling

\[
\begin{align*}
\ddot{x} &= X x, \\
\ddot{z} &= Z z, \\
\dot{t} &= T t.
\end{align*}
\]

Observe that $\max |\dot{x}| = \max |\dot{z}| = U_0$, so it is natural to set

\[
U_0 = \frac{X}{T} = \frac{Z}{T}.
\]

Inserting this into the initial value problem, we get

\[
m \frac{X}{T^2} \begin{bmatrix} \ddot{x} \\
\ddot{z}
\end{bmatrix} = -mg \begin{bmatrix} 0 \\
1
\end{bmatrix} - c \frac{X^2}{T^2} \begin{bmatrix} \dot{x} \\
\dot{z}
\end{bmatrix} \sqrt{\dot{x}^2 + \dot{z}^2},
\]

Gravity dominates, so we balance the first and second terms, giving $X = gT^2 = \frac{U_0^2}{g}$. Inserting this, we end up with the equations from the text, with

\[
\varepsilon = \frac{cU_0^2}{mg}.
\]
Problem 4:

a) The law of mass action gives us that the reaction rate \( r = k \tilde{a} \tilde{b} \). Each reaction produces one molecule of substance A and removes one molecule of substance B, so

\[
\frac{d\tilde{a}}{dt} = r = k \tilde{a} \tilde{b}, \quad \frac{d\tilde{b}}{dt} = -r = -k \tilde{a} \tilde{b}.
\]

Observe that

\[
\frac{d}{dt}(\tilde{a} + \tilde{b}) = 0 \quad \implies \quad \tilde{a} + \tilde{b} = a_0 + b_0,
\]

and hence

\[
\frac{d\tilde{a}}{dt} = k(\tilde{a}(a_0 + b_0 - \tilde{a})).
\]

b) Fick’s law states that the diffusive flux of a substance with concentration \( c \) is

\[
\vec{j}_c = -D \nabla c.
\]

The conservation law for substance A in \( I = [c, d] \) is then

\[
\frac{d}{dt} \int_I \tilde{a} \, dx = -(j_a(d) - j_a(c)) + \int_I k\tilde{a}(M - \tilde{a}) \, dx.
\]

We can transform the integral form to differential form by the standard procedure of setting \( d = c + \Delta x \), dividing by \( \Delta x \) and letting \( \Delta x \) tend to 0. This leads to the PDE

\[
\tilde{a}_t = D\tilde{a}_{\tilde{x}\tilde{x}} + k\tilde{a}(M - \tilde{a}).
\]

If we choose scales \( \tilde{a} = Aa, \tilde{x} = Xx, \tilde{t} = Tt \) with

\[
A = M, \quad T = \frac{1}{ Mk}, \quad X = \sqrt{\frac{D}{Mk}},
\]

and divide by \( A/T \), we end up with the equation in the text.

c) If we insert \( a = 1 + c \) into the scaled PDE and linearize around \( c = 0 \), we get

\[
(c_L)_t = (c_L)_{xx} - c_L.
\]

Letting \( \tilde{c} = e^t c_L \), the equation is reduced to the heat equation

\[
\tilde{c}_t = \tilde{c}_{xx},
\]

which is solved by convolution with the fundamental solution \( c_F \):

\[
\tilde{c} = c_F * \tilde{c}_0 = \int_{-\infty}^{\infty} c_F(x - y, t) \tilde{c}(y, 0) \, dy.
\]
Going back, we get
\[ c_L(x, t) = e^{-t} \tilde{c}(x, t) = e^{-t} \int_{-\infty}^{\infty} c_F(x - y, t) c_L(y, 0) \, dy. \]

Using the hint, we calculate
\[
|c_L| \leq e^{-t} \int_{-\infty}^{\infty} c_F(x - y, t)|c_L(y, 0)| \, dy \leq e^{-t} \max|c_L(y, 0)| \cdot 1 \rightarrow 0 \quad (t \rightarrow \infty).
\]

From this, we conclude that all small perturbations of \( a = 1 \) die out in time, and \( a = 1 \) is an asymptotically stable equilibrium solution.

**Problem 5:**
The red light at \( x = 0, t > 0 \) implies that the flux \( j(\rho) = 0 \) at \( x = 0, t > 0 \), i.e. \( \rho = 0 \) or \( \rho = 1 \) at \( x = 0, t > 0 \). We must choose \( \rho = 0 \) or \( \rho = 1 \) at \( x = 0 \) so that the characteristics go into the domain \( x < 0 \). Since the kinematic velocity is

\[
j'(\rho) = 1 - 2\rho = \begin{cases} 1, & \rho = 0 \\ -1, & \rho = 1, \end{cases}
\]

we must choose \( \rho = 1 \) at \( x = 0 \). At time \( t = 0 \), we are given \( \rho = 1/4 \), so \( j'(\rho) = 1/2 > 0 \). Thus, the characteristics cross and we get a shock solution

\[
\rho(x, t) = \begin{cases} 1, & S(t) \leq x \leq 0 \\ 1/4, & x \leq S(t), \end{cases}
\]

where the shock curve \( S(t) \) satisfies the Rankine–Hugoniot condition

\[
\dot{S}(t) = \frac{j(\rho_{\text{left}}) - j(\rho_{\text{right}})}{\rho_{\text{left}} - \rho_{\text{right}}} = \frac{j(1/4) - j(1)}{1/4 - 1} = -1/4, \quad S(0) = 0,
\]

so \( S(t) = -t/4 \).