EXAM IN TMA4205 NUMERICAL LINEAR ALGEBRA

Friday December 8, 2006
Time: 09:00–13:00

Aids: A – Alle printed and hand written aids are allowed.
     All calculators are allowed.

Problem 1

a) Show that the inverse of the matrix

\[ I - uv^T \]

where \( I \) is the \( n \times n \) identity matrix and \( u, v \in \mathbb{R}^n \) and \( v^T u \neq 1 \), is of the type

\[ I + \gamma uv^T. \]

Find \( \gamma \).

b) Estimate the condition number \( K_2(I - uv^T) \) by using \( \|u\|_2 \) and \( \|v\|_2 \). Suppose \( v^T u \neq 1 \).

c) Suppose that we are solving the \( n \times n \) linear system

\[ Ax = b, \quad A = B - wz^T, \quad z \in \mathbb{R}^n, \quad w = Bz \]

where \( B \) stems from the discretization of a Laplacian, for instance, by the finite difference or the finite element method. Suppose that \( n \) is large, that \( B \) is invertible, and that we can use a multigrid V- or W-cycle for efficient solution of linear systems of the form \( By = c \). We shall therefore use \( B^{-1} \) as preconditioner in our problem.
Find the conditions that $z$ must fulfill in order to guarantee the convergence of the conjugate gradient method when applied to the preconditioned system

$$B^{-1}Ax = B^{-1}b.$$ 

Suppose that $\|z\|_2 \leq 0.5$, and use the convergence estimate for the conjugate gradient algorithm and the estimate of $K_2(I - zz^T)$ for finding the minimal number of iterations necessary to guarantee that

$$\frac{\|x - x_m\|_{B^{-1}A}}{\|x - x_0\|_{B^{-1}A}} \leq 10^{-3}.$$ 

d) Use the result from a) and find an algorithm for solving $Ax = b$ that works for every $z$ and $w$ such that $A$ is invertible.

e) Suppose that $B$ is an $n \times n$ matrix that stems from a discretization of the Helmholtz equation with periodic boundary conditions, i.e.

$$\alpha u(x) + \Delta u(x) = \psi(x), \quad -\pi \leq x \leq \pi, \quad u(-\pi) = u(\pi), \quad 0 < \alpha \leq 1,$$

where $\Delta$ is the Laplacian. After discretization with the spectral method we get

$$B = \hat{\Omega}^H \Lambda \hat{\Omega}, \quad \hat{\Omega}^H \hat{\Omega} = I,$$

where $\Lambda$ is a diagonal matrix. We suppose that $n$ is an even integer. The diagonal of $\Lambda$ is

$$[\alpha, \alpha, \alpha + 1, \alpha + 1, \ldots, \alpha + (k - 1)^2, \alpha + (k - 1)^2, \alpha + k^2, \alpha + k^2],$$

with $k = n/2 - 1$. The unitary matrix $\hat{\Omega}$ is such that $\hat{\Omega} = P \Omega P^T$ where $P$ is a permutation matrix, and $\Omega$ is the Fourier matrix. Show that the diagonal elements in the matrix $\hat{B}$ is

$$B_{j,j} = \frac{2 \cdot \alpha}{n} + \frac{n^2 - 3n + 2}{12}, \quad j = 1, \ldots, n.$$

**Hint.** Note that the matrix $\hat{B} = \Omega^H (P^T \Lambda P) \Omega$ is cyclic, symmetric. Find the diagonal elements of $\hat{B}$. Show that $\hat{B}$ and $B$ have the same diagonal elements.

**Given.** A permutation matrix is a matrix obtained by permuting the rows or columns of the identity matrix.

The Fourier matrix $\Omega$ has elements

$$\Omega_{p,l} = \frac{1}{\sqrt{n}} \exp \left( i \cdot \frac{2\pi}{n} (p-1)(l-1) \right), \quad i = \sqrt{-1}, \quad p, l = 1, \ldots, n.$$ 

The eigenvalues of a cyclic matrix are the components of the vector

$$g = \sqrt{n} \cdot \Omega^H \hat{b},$$

where $\hat{b}$ is the first row in $\hat{B}$.

Remember that

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6}.$$
f) Consider the weighted Jacobi-iteration for solving $By = c$, d.v.s.

$$y^{m+1} = (1 - \omega)y^m + \omega D^{-1}(E + F)y^m + \omega D^{-1}c,$$

where $B = D - E - F$ where $D$ is diagonal and $E$ is lower triangular, and $F$ upper triangular, and $0 < \omega \leq 1$ is the relaxation parameter.

Show that the iteration can be written as

$$y^{m+1} = G_\omega y^m + \omega D^{-1}c,$$

and use this to show that the eigenvalues of $G_\omega$ are

$$\mu_j = 1 - \omega \frac{12 \cdot \lambda_j}{n^2 - 3n + 2 + 24\alpha/n}, \quad j = 1, \ldots, n$$

where $\lambda_j$ are the eigenvalues of $B$, i.e.

$$\lambda_j = \begin{cases} 
\alpha + \left(\frac{j}{2} - 1\right)^2, & \text{if } j \text{ is even,} \\
\alpha + \left(\frac{j+1}{2} - 1\right)^2, & \text{if } j \text{ is odd.}
\end{cases}$$

g) Investigate the smoothing properties of weighted Jacobi. Express the initial error $e^0 = y - y^0$ as

$$e^0 = \sum_{j=1}^n f_j w_j,$$

where $w_j$ are the columns of the matrix $\tilde{\Omega}$. Find the corresponding formula for the error $e^m = y - y^m$ by using the coefficients $f_j$ and the eigenvalues and eigenvectors of $G_\omega$. Determine $\omega$ which yields the best damping of the high frequency error modes from the condition $-\mu_{n/2} = \mu_n$.

**Problem 2**

We shall consider sensitivity with respect to rounding error in the system $AXC = B$ where $A$ is a real $n \times n$ invertible matrix, $X$ is a real $n \times p$ matrix, $C$ is a real $p \times p$ invertible matrix and $B$ is an $n \times p$ matrix, with $n \geq p$. Consider the perturbed system

$$(A + \varepsilon \Delta A)X(\varepsilon)(C + \varepsilon \Delta C) = B + \varepsilon \Delta B.$$ 

Find an upper bound for the relative error,

$$\frac{\|X(\varepsilon) - X\|_2}{\|X\|_2},$$

by using the relative error in input data, $A$, $B$ og $C$ and the condition numbers of $A$ and $C$. 
Problem 3
Consider the Arnoldi algorithms for computing an orthonormal basis of the Krylov subspace

\[K_m(A, u_0) = \text{span}\{u_0, Au_0, \ldots, A^{m-1}u_0\},\]

where \(A\) is an \(n \times n\) matrix and \(u_0 \in \mathbb{R}^n\). The eigenvalues of the Arnoldi upper Hessenberg matrix, \(V_m^TAV_m = H_m\), can be computed efficiently for instance by using a shifted QR iteration algorithm. Explain why. Suppose that \(\nu_k\) is an eigenvalue of \(H_m\) and \(y_k \in \mathbb{R}^m\) is the corresponding normalized eigenvector. Consider \(\nu_k\) as an approximation to an eigenvalue of \(A\), and \(V_my_k\) as an approximation to the corresponding eigenvector. Find an error bound for

\[\|AV_my_k - \nu_kV_my_k\|_2.\]

Use known properties of the Arnoldi algorithm for this purpose.