Problem 1  

The partial differential equation

\[-u_{xx} + cu = f, \quad c \geq 0.\]

with homogeneous Dirichlet boundary conditions, yields after discretizing with centered differences, a linear system of the form \(Au = b\) where \(A = \text{tridiag}(-1, 2 + \gamma, -1), \ A \in \mathbb{R}^{m \times m}\), \(\gamma = c/(m + 1)^2\). We find that \(A\) has eigenvalues

\[
\lambda_k = \gamma + 4 \sin^2 \left( \frac{k\pi}{2(m+1)} \right), \quad k = 1, \ldots, m,
\]

and corresponding eigenvectors

\[
w_k = \begin{bmatrix}
\sin \left( \frac{k\pi}{m+1} \right) \\
\sin \left( \frac{2k\pi}{m+1} \right) \\
\vdots \\
\sin \left( \frac{mk\pi}{m+1} \right)
\end{bmatrix}.
\]

a) Formulate the weighted Jacobi method with relaxation parameter \(\omega\) for this linear system, and show that the iteration can be written in the form

\[
u^{(q+1)} = G_\omega v^{(q)} + \frac{\omega}{2 + \gamma} b \quad \text{where} \quad G_\omega = I - \frac{\omega}{2 + \gamma} A,
\]
and that the iteration matrix $G_\omega$ has eigenvalues

$$
\mu_k = 1 - \frac{\omega}{2 + \gamma} \left( \gamma + 4 \sin^2 \left( \frac{k\pi}{2(m+1)} \right) \right), \ k = 1, \ldots, m,
$$

and the same eigenvectors as $A$.

b) The error after $q$ iterations with weighted Jacobi on this system can be written as

$$
e^{(q)} = G_\omega^q e^{(0)} = \sum_{k=1}^{m} \rho_k \mu_k^q w_k,
$$

where

$$
e^{(0)} = \sum_{k=1}^{m} \rho_k w_k
$$

is the initial error $e^{(0)}$ expressed in terms of the eigenvectors $w_k$.

Determine the (optimal) value $\omega^{opt}$ of $\omega$ from which the best damping occurs of the upper half of the spectrum of the error, i.e. find

$$
\omega^{opt} = \arg \min \omega \max_{k > \frac{m+1}{2}} |\mu_k|.
$$

Verify that you get back to the known $\omega^{opt} = 2/3$ when $\gamma = 0$. What happens when $\gamma$ tends to infinity?

Problem 2

a) Describe briefly the idea behind the projection methods for solving linear systems $Ax = b$.

Use approximately 4-5 lines.

Assume in the rest of this problem that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $b \in \mathbb{R}^{n}$ is a given right hand side, and $x = A^{-1}b$. We let $r_k = b - Ax_k$ and $e_k = x - x_k = A^{-1}r_k$, for $k \geq 0$ and assume that $x_0$ is a given vector.

b) Let us use as approximation space $K = \text{span} \{v\}$ and constraint space $L = K$. Let $x_1$ be the result of one step with the projection method. Show that

$$
\langle Ae_1, e_1 \rangle = \langle Ae_0, e_0 \rangle - \langle r_0, v \rangle^2 / \langle Av, v \rangle.
$$
c) The method from the previous question says nothing about how the new search direction \( v \) is chosen in each iteration. Let us therefore introduce the following principle: Choose \( v_k \) such that \( \langle v_k, r_k \rangle = \|r_k\|_1 \), that is, let the components in \( v_k \) be 1 and \(-1\), negative \((-1)\) if the corresponding component in \( r_k \) is negative, and positive \((+1)\) if the \( r_k \) component is \( \geq 0 \). Show that
\[
\|e_{k+1}\|_A \leq \left(1 - \frac{1}{n\kappa(A)}\right)^{1/2} \|e_k\|_A.
\]
Here \( \kappa(A) = \|A\|_2\|A^{-1}\|_2 \) while \( \|w\|_A = \langle Aw, w \rangle^{1/2} \).

**Problem 3**  
Let the matrix \( A \) be given as
\[
A = \begin{bmatrix}
8 & 1 & 0 \\
1 & 4 & \varepsilon \\
0 & \varepsilon & 1
\end{bmatrix}, \quad |\varepsilon| \leq 1.
\]

a) Give an estimate for the eigenvalues of \( A \) by using the Gerschgorin theorem. In particular, what can one say about the smallest eigenvalue? Make a sketch to illustrate.

b) Show for instance by using a suitable diagonal similarity transformation the sharper estimate \( |\lambda_3 - 1| \leq \varepsilon^2 \) for the smallest eigenvalue of \( A \).

c) For \( \varepsilon = 0.1 \) one has found \( Q \) and \( R \) such that \( A - I = QR \) where
\[
Q = \begin{bmatrix}
-0.9899 & 0.1413 & 0.0050 \\
-0.1414 & -0.9893 & -0.0350 \\
0 & -0.0353 & 0.9994
\end{bmatrix}, \quad R = \begin{bmatrix}
-7.0711 & -1.4142 & -0.0141 \\
0 & -2.8302 & -0.0989 \\
0 & 0 & -0.0035
\end{bmatrix}.
\]

Find an approximation to the smallest eigenvalue of \( A \) from this.

**Problem 4**

a) Find the singular value decomposition of the matrix
\[
A = \begin{bmatrix}
1 & 0 \\
-1 & 1 \\
0 & -1
\end{bmatrix}.
\]

b) The matrix in the previous question is a special case of a matrix \( B \in \mathbb{R}^{(n+1) \times n} \) where \( B_{k,k} = 1, \ B_{k+1,k} = -1 \) for \( k = 1, \ldots, n \) and where all other elements of \( B \) are zero. Determine the singular value decomposition of \( B \).