Problem 1  The partial differential equation

\[-u_{xx} + cu = f, \quad c \geq 0.\]

with homogeneous Dirichlet boundary conditions, yields after discretizing with centered differences, a linear system of the form \(Au = b\) where \(A = \text{tridiag}(-1, 2 + \gamma, -1)\), \(A \in \mathbb{R}^{m \times m}\), \(\gamma = c/(m + 1)^2\). We find that \(A\) has eigenvalues

\[\lambda_k = \gamma + 4\sin^2\left(\frac{k\pi}{2(m + 1)}\right), \quad k = 1, \ldots, m,\]

and corresponding eigenvectors

\[w_k = \begin{bmatrix} \sin \left(\frac{k\pi}{m+1}\right) \\
\sin \left(\frac{2k\pi}{m+1}\right) \\
\vdots \\
\sin \left(\frac{mk\pi}{m+1}\right) \end{bmatrix}.\]

a) Formulate the weighted Jacobi method with relaxation parameter \(\omega\) for this linear system, and show that the iteration can be written in the form

\[u^{(q+1)} = G_\omega u^{(q)} + \frac{\omega}{2 + \gamma} b \quad \text{where} \quad G_\omega = I - \frac{\omega}{2 + \gamma} A,
\]
and that the iteration matrix $G_\omega$ has eigenvalues

$$
\mu_k = 1 - \frac{\omega}{2 + \gamma} \left( \gamma + 4 \sin^2 \left( \frac{k\pi}{2(m+1)} \right) \right), \quad k = 1, \ldots, m,
$$

and the same eigenvectors as $A$

**Answer:** Standard Jacobi results from the splitting $A = M - N = 2 + \gamma I - ((2 + \gamma)I - A)$ and thus

$$(2 + \gamma) u^{(q+1)} = ((2 + \gamma)I - A)u^{(q)} + b$$

$G_\omega$ is obtained by dividing by $(2 + \gamma)$ on each side. Eigenvalues and eigenvectors are found from

$$G_\omega w_k = (I - \frac{\omega}{2 + \gamma} A)w_k = \left(1 - \frac{\omega}{2 + \gamma} \lambda_k \right) w_k$$

where we insert the given $\lambda_k$.

b) The error after $q$ iterations with weighted Jacobi on this system can be written as

$$e^{(q)} = G_\omega^q e^{(0)} = \sum_{k=1}^{m} \rho_k \mu_k^q w_k,$$

where

$$e^{(0)} = \sum_{k=1}^{m} \rho_k w_k$$

is the initial error $e^{(0)}$ expressed in terms of the eigenvectors $w_k$.

Determine the (optimal) value $\omega^\text{opt}$ of $\omega$ from which the best damping occurs of the upper half of the spectrum of the error, i.e. find

$$\omega^\text{opt} = \arg \min_{\omega} \max_{k > m+1} |\mu_k|.$$

Verify that you get back to the known $\omega^\text{opt} = 2/3$ when $\gamma = 0$. What happens when $\gamma$ tends to infinity?

**Answer:** We introduce the variable $\theta = k/(m+1)$, and find that

$$\mu(\theta) = 1 - \frac{\omega}{2 + \gamma} \left( \gamma + 4 \sin^2 \left( \frac{\pi}{2} \theta \right) \right)$$

$\mu(\theta)$ is a decreasing function of $\theta$, so min and max are attained in the end points of the closed interval $\theta \in [1/2, 1]$. We conclude that the optimal $\omega$ is obtained when $\mu(1/2) = -\mu(1)$, that is

$$1 - \frac{\omega(\gamma + 2)}{\gamma + 2} = - \left(1 - \frac{\omega(\gamma + 4)}{\gamma + 2} \right)$$

which yields

$$\omega^\text{opt} = \frac{\gamma + 2}{\gamma + 3},$$

so $\gamma = 0$ leads to the well-known result $\omega^\text{opt} = 2/3$. When $\gamma$ increases, $\omega^\text{opt}$ tends to the value 1.
Problem 2

a) Describe briefly the idea behind the projection methods for solving linear systems $Ax = b$. Use approximately 4-5 lines.

**Answer:** The idea is to define an approximation space $K$ and a constraint space $L$ of the same dimension, and thereafter seek, for a given $x_0$, a vector $x$ such that

$$x - x_0 \in K, \quad b - Ax \perp L$$

Assume in the rest of this problem that $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, $b \in \mathbb{R}^n$ is a given right hand side, and $x = A^{-1}b$. We let $r_k = b - Ax_k$ and $e_k = x - x_k = A^{-1}r_k$, for $k \geq 0$ and assume that $x_0$ is a given vector.

b) Let us use as approximation space $K = \text{span} \{v\}$ and constraint space $L = K$. Let $x_1$ be the result of one step with the projection method. Show that

$$\langle Ae_1, e_1 \rangle = \langle Ae_0, e_0 \rangle - \langle r_0, v \rangle^2 / \langle Av, v \rangle$$

**Answer:** The method is

$$x_1 = x_0 + \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} v \Rightarrow e_1 = e_0 - \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} v$$

Since $r_1 \perp v$

$$\langle Ae_1, e_1 \rangle = \langle r_1, e_1 \rangle = \langle r_1, e_0 \rangle = \langle Ae_0, e_0 \rangle - \frac{\langle r_0, v \rangle}{\langle Av, v \rangle} \langle Av, e_0 \rangle = \langle Ae_0, e_0 \rangle - \frac{\langle r_0, v \rangle^2}{\langle Av, v \rangle}$$

In the last step we have used that $A$ is symmetric.

c) The method from the previous question says nothing about how the new search direction $v$ is chosen in each iteration. Let us therefore introduce the following principle: Choose $v_k$ such that $\langle v_k, r_k \rangle = \|r_k\|_1$, that is, let the components in $v_k$ be 1 and $-1$, negative ($-1$) if the corresponding component in $r_k$ is negative, and positive ($+1$) if the $r_k$ component is $\geq 0$. Show that

$$\|e_{k+1}\|_A \leq \left(1 - \frac{1}{\kappa(A)}\right)^{1/2} \|e_k\|_A.$$ 

Here $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$ while $\|w\|_A = \langle Aw, w \rangle^{1/2}$.

**Answer:** Rewriting the result from the previous question yields

$$\|e_{k+1}\|_A^2 = \left(1 - \frac{\langle r_k, v_k \rangle^2}{\langle Av_k, v_k \rangle \langle Ae_k, e_k \rangle}\right) \|e_k\|_A^2.$$

Now use \((r_k, v_k) = \|r_k\|_1 \geq \|r_k\|_2\), and that
\[
\langle Av_k, v_k \rangle (Ae_k, e_k) = \langle Av_k, v_k \rangle (r_k, A^{-1}r_k) \leq \|A\|_2 \|A^{-1}\|_2 \|v_k\|_2^2 \|r_k\|_2^2 = n \kappa(A) \|r_0\|_2^2
\]
We thereby get
\[
\frac{\langle r_k, v_k \rangle^2}{\langle Av_k, v_k \rangle(Ae_k, e_k)} \geq \frac{\|r_k\|_2^2}{n \kappa(A) \|r_k\|_2^2} = \frac{1}{n \kappa(A)}
\]
and as desired we obtain
\[
\|e_{k+1}\|_A^2 \leq \left(1 - \frac{1}{n \kappa(A)}\right) \|e_k\|_A^2
\]

Problem 3 Let the matrix \(A\) be given as
\[
A = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \quad |\varepsilon| \leq 1.
\]

a) Give an estimate for the eigenvalues of \(A\) by using the Gershgorin theorem. In particular, what can one say about the smallest eigenvalue? Make a sketch to illustrate.

Answer: All three Gerschgorin circles are disjoint, and the matrix is symmetric so the eigenvalues are real. We find that the eigenvalues \(\lambda_1, \lambda_2, \lambda_3\) satisfy
\[
7 \leq \lambda_1 \leq 9, \quad 3 - \varepsilon \leq \lambda_2 \leq 5 + \varepsilon, \quad 1 - \varepsilon \leq \lambda_3 \leq 1 + \varepsilon.
\]
In particular, the smallest eigenvalue is between \(1 - \varepsilon\) and \(1 + \varepsilon\).

b) Show for instance by using a suitable diagonal similarity transformation the sharper estimate \(|\lambda_3 - 1| \leq \varepsilon^2\) for the smallest eigenvalue of \(A\).

Answer: We compute the similarity transformation with \(T = \text{diag}(1, 1, \varepsilon)\) and
\[
TAT^{-1} = \begin{bmatrix} 8 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & \varepsilon^2 & 1 \end{bmatrix}
\]
and the third Gerschgorin disk is still disjoint from the others whenever \(|\varepsilon| \leq 1\), and therefore contains an eigenvalue.

c) For \(\varepsilon = 0.1\) one has found \(Q\) and \(R\) such that \(A - I = QR\) where
\[
Q = \begin{bmatrix} -0.9899 & 0.1413 & 0.0050 \\ -0.1414 & -0.9893 & -0.0350 \\ 0 & -0.0353 & 0.9994 \end{bmatrix}, \quad R = \begin{bmatrix} -7.0711 & -1.4142 & -0.0141 \\ 0 & -2.8302 & -0.0989 \\ 0 & 0 & -0.0035 \end{bmatrix}.
\]
Find an approximation to the smallest eigenvalue of $A$ from this.

**Answer:** We can easily perform one iteration with the shifted QR method, we just need to determine the $(3,3)$ element in $A_1 = RQ + I$ which becomes

$$1 + (-0.0035) \cdot 0.9994 = 0.9965,$$

The answer is correct in all 4 digits.

**Problem 4**

**a)** Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

**Answer:**

$$\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$$

**b)** The matrix in the previous question is a special case of a matrix $B \in \mathbb{R}^{(n+1) \times n}$ where $B_{k,k} = 1$, $B_{k+1,k} = -1$ for $k = 1, \ldots, n$ and where all other elements of $B$ are zero. Determine the singular value decomposition of $B$.

**Answer:** We must find $U, V, \Sigma$ such that $B = U \Sigma V^T$. Here we find that $B^T B = \text{tridiag}(-1, 2, -1)$ for which we know the eigenvalues:

$$\sigma_k^2 = 4 \sin^2 \left( \frac{k\pi}{2(n+1)} \right) \implies \sigma_k = 2 \sin \left( \frac{k\pi}{2(n+1)} \right), \quad k = 1, \ldots, n.$$

The matrix $V$ has the eigenvectors of $B^T B$ as columns, they are also known, but we need to scale them such that the Euclidian norm is 1.

$$\|w_k\|_2^2 = \sum_{j=1}^{n} \sin^2 \left( \frac{jk\pi}{2(n+1)} \right) = \frac{n+1}{2}$$

So column $k$ in $V$ is

$$v_k = \sqrt{\frac{2}{n+1}} \begin{bmatrix} \sin \left( \frac{k\pi}{n+1} \right) \\ \sin \left( \frac{2k\pi}{n+1} \right) \\ \vdots \\ \sin \left( \frac{nk\pi}{n+1} \right) \end{bmatrix}.$$
To find column $k$ in $U$, we set $u_k = \frac{1}{\sigma_k} B v_k$. Component $\ell$ of $u_k$ is therefore (at the moment we ignore the factor $\sqrt{2/(n+1)}$)

$$
\frac{1}{\sigma_k} \left( \sin \left( \frac{k\ell \pi}{n+1} \right) - \sin \left( \frac{k(\ell - 1) \pi}{n+1} \right) \right)
$$

Here we could have called it a day, but it is in fact here all the fun begins. Letting

$$
\phi = \frac{k(\ell - 1/2) \pi}{n+1}, \quad \delta = \frac{k\pi/2}{n+1} \Rightarrow \sigma_k = 2 \sin \delta
$$

the above expression becomes

$$
\frac{1}{\sigma_k} (\sin(\phi + \delta) - \sin(\phi - \delta)) = \cos \phi \frac{2 \sin \delta}{\sigma_k} = \cos \phi = \cos \left( \frac{k(\ell - 1/2) \pi}{n+1} \right)
$$

We therefore have the following elegant expression for column $k$ in $U$

$$
{u_k} = \sqrt{\frac{2}{n+1}} \begin{bmatrix}
\cos \left( \frac{k\frac{1}{2} \pi}{n+1} \right) \\
\cos \left( \frac{k\frac{1}{2} \pi}{n+1} \right) \\
\vdots \\
\cos \left( \frac{k(\frac{n+1}{2}) \pi}{n+1} \right)
\end{bmatrix} \in \mathbb{R}^{n+1}.
$$