Problem 1  The real nonsingular matrix
\[
A = \begin{bmatrix}
2 & 0 & 0 & 1 \\
0 & 3 & -1 & 1 \\
0 & -1 & 2 & 0 \\
1 & 1 & 0 & 2
\end{bmatrix}
\]
is unitarily similar to an upper Hessenberg matrix $H$.

a) Show that $H$ is also tridiagonal.

b) Is $H$ positive-definite?
   [Hint: Use Gerschgorin’s theorem to investigate bounds for the eigenvalues].

c) Use Householder reflections to find a unitary matrix $Q$ so that $H = Q^T AQ$.
   [You may use the information given at the footnote on the last page].

Problem 2

a) Consider a matrix of the form
\[
A = I + \mu S,
\]
where $\mu$ is a scalar and $S$ is skew-symmetric, and $I$ is the identity matrix.

i) Show that $A$ is positive definite.
ii) Consider the Arnoldi process for $A$. Show that the resulting Hessenberg matrix will have the following tridiagonal form:

$$H_m = \begin{bmatrix}
1 & -\eta_2 & & \\
\eta_2 & 1 & -\eta_3 & \\
& \ddots & \ddots & \ddots \\
\eta_{m-1} & & 1 & -\eta_m \\
\eta_m & & & 1
\end{bmatrix}.$$ 

b) Suppose $Ax = b$ is a linear system where $A$ is symmetric and positive-definite. Consider an orthogonal projection method for which the search and constraint spaces are given by $L = K = \text{span} \{r_0, Ar_0\}$ where $r_0 = b - Ax_0$ is the current residual. Let $\tilde{x} = x_0 + \alpha r_0 + \beta Ar_0$ denote the solution update, where $\alpha$ and $\beta$ are some real constants. Prove the error satisfies:

$$\|\tilde{e}\|_A^2 = \left[ 1 - \alpha \frac{(r_0, r_0)}{(A^{-1}r_0, r_0)} - \beta \frac{(Ar_0, r_0)}{(A^{-1}r_0, r_0)} \right] \|e_0\|_A^2$$

where $\tilde{e} = x - \tilde{x}$, $e_0 = x - x_0$, $\|y\|_A = \sqrt{(Ay, y)}$. Derive the expressions for $\alpha$ and $\beta$. Find an expression for the lower bound of the term $\beta \frac{(Ar_0, r_0)}{(A^{-1}r_0, r_0)}$. The answer must be expressed only in terms of $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$, the minimum and maximum eigenvalues of $A$ respectively. [Hint: The spectrum of an SPD matrix provides a bound for its Rayleigh quotients].

**Problem 3** In a finite difference discretization of the one-dimensional Poisson equation

$$-u_{xx} = f,$$ in $\Omega = (0, 1)$,

$$u(0) = u(1) = 0,$$ 

on a uniform grid we obtain a linear system $Ax = b$ where $A = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{(n-1) \times (n-1)}$, $x \in \mathbb{R}^{n-1}$ is the vector of unknowns, and $h = \frac{1}{n}$ is the mesh parameter. The eigenvalues of $A$ are known to be given by

$$\lambda_m = 2 - 2 \cos \left( \frac{m\pi}{n} \right), \quad m = 1, \ldots, n - 1.$$ 

The symmetric successive overrelaxation (SSOR) method for solving the system $Ax = b$ is an iterative method that can be described in the following two steps

$$(D - \omega E)x_{k+1/2} = [\omega F + (1 - \omega)D]x_k + \omega b,$$

$$(D - \omega F)x_{k+1} = [\omega E + (1 - \omega)D]x_{k+1/2} + \omega b,$$

$k = 0, 1, \ldots$, where $\omega$ is the relaxation parameter, and $D$ is the diagonal part of $A$, while $E$ and $F$ define, respectively, the lower and the upper triangular parts of $A$ so that $A = D - E - F$. The iteration matrix is defined as a matrix $G$ so that the iterative method can be expressed as a fixed point iteration $x_{k+1} = Gx_k + f$.

a) Prove that the iteration matrix for the SSOR method can be expressed as

$$G_\omega = I - \omega(2 - \omega)(D - \omega F)^{-1}D(D - \omega E)^{-1}A,$$ 

where $I$ is the identity matrix.
b) To use the SSOR as preconditioning for the conjugate gradient method we consider a splitting of the form \( A = M - N \), so that the SSOR iteration takes the form

\[
x_{k+1} = M^{-1}N x_k + M^{-1}b = x_k + M^{-1}r_k,
\]

where \( r_k \) is the residual vector, and \( G = M^{-1}N \) is the iteration matrix. The matrix \( M \) is the required preconditioner. Assume that \( \omega \in (0, 2) \). Deduce from \( G_\omega \) an expression for the preconditioner \( M = M_\omega \), and show that it is symmetric and positive-definite. [Hint: In this example \( F = E^T \)].

c) For \( \omega \in (0, 2) \) the matrix \( M_\omega^{-1}A \) of the preconditioned system is known to have condition number \( \frac{2a n^2 + \lambda_{\min}}{(2 - \omega) \lambda_{\min}} \), where \( a = \frac{(2 - \omega)^2}{4\omega} \) and \( \lambda_{\min} \) is the minimum eigenvalue of \( A \). Show that the optimal value of \( \omega \) for the preconditioner \( M_\omega \) is given by

\[
\omega_{\text{opt}} = \frac{2\sqrt{\gamma}}{1 + \sqrt{\gamma}}
\]

where \( \gamma = \frac{n^2}{\lambda_{\min}} \). Assume that \( n \to \infty \). How does the condition number of \( M_\omega^{-1}A \) varies as a function of \( n \) when \( \omega = \omega_{\text{opt}} \)?

**Problem 4**  Consider the matrix

\[
A = \begin{bmatrix}
-1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1 \\
1 & 0 & 1
\end{bmatrix}.
\]

a) Under which conditions can the normal equation \( A^T A x = A^T b \) be solved using conjugate gradient iterations? Are these conditions satisfied in this case?

b) Find the condition number of \( A^T A \) in the 2-norm. Would you expect any improvement in the conjugate gradient method by using diagonal preconditioning?

c) Calculate the singular values of \( A \).

d) Find the 1-, 2-, \( \infty \)- and Frobenius-norms of \( A \).
Some useful information:

**For problem 1.** In the Householder reduction into Hessenberg form, the first Householder reflector applied to the left and right of $A$, reduces it to the matrix

$$
\begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & 0 & 1 \\
0 & 0 & 2 & 1 \\
0 & 1 & 1 & 3
\end{bmatrix}.
$$

**Kantorovich inequality.** Suppose $B \in \mathbb{R}^{n \times n}$ is a symmetric and positive definite matrix. Then

$$
\frac{(Bx, x)(B^{-1}x, x)}{(x, x)^2} \leq \frac{(\lambda_{\text{max}} + \lambda_{\text{min}})^2}{4\lambda_{\text{max}}\lambda_{\text{min}}},
$$

for all $x \in \mathbb{R}^n$, $x \neq 0$, where $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ represent the smallest and largest eigenvalues of $B$ respectively.

**Condition number.** The condition number of a matrix $A$ is given by the formula $\kappa(A) = \|A\|\|A^{-1}\|$. In the $p$-norm, this is given by $\kappa_p(A) = \|A\|_p\|A^{-1}\|_p$. 