EXAMINATION IN NUMERICAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENCE METHODS

MONDAY, JUNE 10, 2005
TIME 09:00-12:00
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Exercisie 1. We consider the problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

in the domain $x \in [0,1]$ and $t \geq 0$. We also have that

$$u(1,t) = \phi_0(t) \text{ and } u(x,0) = g(t),$$

and we assume that (1) is well-posed. Set $\Delta x = \frac{1}{m+1}$ with $m \geq 0$. We solve this differential equation using the $\theta-$method with $\Delta t > 0$. Let the numerical values at (x_l, t_n) be U_l^n. The $\theta-$method can be written as

$$U_l^{n+1} = U_l^n + \theta\mu(U_{l-1}^n - 2U_l^n + U_{l+1}^n) + (1-\theta)\mu(U_{l-1}^{n+1} - 2U_l^{n+1} + U_{l+1}^{n+1}) \tag{2}$$

with Courant number $\mu = \frac{\Delta t}{(\Delta x)^2}$ and $0 \leq \theta \leq 1$.

a) Let

$$U^n = [U_1^n, \ldots, U_m^n]^T.$$

Find $M \in \mathbf{R}^{m\times m}$, $A \in \mathbf{R}^{m\times m}$ and $F \in \mathbf{R}^m$ such that

$$MU^{n+1} = AU^n + F \tag{3}$$

where F comes from the boundary contributions.

b) Show that M in (3) is regular.

c) Let $u_l^n = u(x_l, t_n)$. The truncation error T_l^{n+1} of the $\theta-$method is defined as

$$T_l^{n+1} = u_l^{n+1} - u_l^n - (1-\theta)\mu(u_{l-1}^{n+1} - 2u_l^{n+1} + u_{l+1}^{n+1}) - \theta\mu(u_{l-1}^n - 2u_l^n + u_{l+1}^n).$$

Show that

$$|T_l^{n+1}| \leq C(\Delta x)^4$$

where C is a constant and μ is constant. You can assume that exact solution $u(x,t)$ is at least four times differentiable in space and at least two times differentiable in time.

Hint 1: $u(x_{l-1}, t_n) - 2u(x_l, t_n) + u(x_{l+1}, t_n) = (\Delta x)^2 u_{xx}(x_l, t_{n+1}) + \mathcal{O}(\Delta x)^4$.

Hint 2: Use $\Delta t = \mu(\Delta x)^2$, when deciding which term to drop out of the expression.

d) Show that the $\theta-$method is convergent without using Lax's equivalence theorem.

Hint 1: Remember to generalize $\mu \leq 1/2$ in the Euler case to the $\theta-$method.

Hint 2: Define e_l^n as the error at the point (x_l, t_n) and

$$\eta^n = \max_l |e_l^n|.$$