EXAMINATION IN NUMERICAL SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS USING DIFFERENCE METHODS

MONDAY, JUNE 4, 2007
TIME 09:00-13:00
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Exercisie 1. Given the initial-boundary value problem

$$\begin{aligned} u_t = &\ \partial_x(a(x)\partial_x u), \quad 0 < x < 1,\ t > 0, \\ u(x,0) = &\ f(x), \quad 0 < x < 1, \\ u(0,t) = &\ u(1,t) = 0, \quad t \geq 0, \end{aligned}$$

where $a(x)$ is continuous and positive in the interval $[0,1]$.

a) Discretize this equation using centered differences in space, replacing ∂_x with $(1/h)\delta_x$, and using Eulers method in time. Define the vector $U^n = [U_1^n, \ldots, U_M^n]^T, h = 1/(M+1)$, and show that that the difference method satisfies a recurrence relation of the type

$$U^{n+1} = CU^n, \quad C \in \mathbf{R}^{M\times M}.$$

Determine the matrix C.

b) Let $r = k/h^2$ where k is the time step, $\alpha = \max_{0\leq x\leq 1} a(x)$. Show that the method is stable for stepsizes satisfying

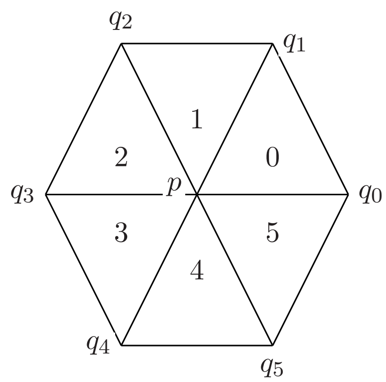
$$2\alpha r \leq 1.$$

Hint. Gershgorin's theorem.

Exercise 2. In this exercise we study different aspects of the finite element method applied to the Poisson problem with homogeneous boundary conditions

$$-\Delta u = f\ (x,y) \in \Omega, \quad u = 0\ (x,y) \in \partial\Omega. \tag{1}$$

Assume that Ω can be partitioned into a uniform triangulation where each edge has length h. A description of the triangulation is given in the figure (on the right), where we assume $q_0, \ldots, q_5$ are interior nodes. The 6 traingles T_j are marked on the figure by the corresponding indices j. We introduce a finite element space $S_h \subseteq S$ as a subspace of S consisting of functions that are linear on each triangle. The basis for S_h are pyramid functions $\{\phi_p(x,y) : p \text{ interior node in } \Omega\}$. For the outline of the figure, we define a local coordinate system by setting



$$\xi = \frac{x - x_p}{h}, \quad \eta = \frac{y - y_p}{h},$$

so that the edges joining p to the vertices $(\pm 1, 0)$ and $(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$ of the hexagon are normalized. We define the vector $\mathbf{z} = (\xi, \eta)$.

a) Shape functions $\psi_p^j,\ j = 0, \ldots, 5$, are basis functions ϕ_p restricted to triangles T_j (see figure). Derive the expressions for $\psi_p^0(\mathbf{z}),\ \psi_{q_0}^0(\mathbf{z})$ and $\psi_{q_1}^0(\mathbf{z})$.

b) Derive a $2 \times 2-$matrix Q such that

$$\begin{aligned}\psi_p^{j+1}(\mathbf{z}) &= \psi_p^j(Q\mathbf{z}) \\ \psi_{q_{j+1}}^{j+1}(\mathbf{z}) &= \psi_{q_j}^j(Q\mathbf{z}) \\ \psi_{q_{j+1}}^j(\mathbf{z}) &= \psi_{q_j}^{j-1}(Q\mathbf{z}),\end{aligned}$$

where, by definition, we set $\psi_p^6(\mathbf{z}) := \psi_p^0(\mathbf{z})$ and $\psi_{q6}^j := \psi_{q0}^j$. Determine Q and verify that it is orthogonal ($Q^TQ = I$) with determinant 1.

c) Hence, verify that

$$\begin{aligned}\int_{T_j} |\nabla\psi_p^j|^2 d\xi d\eta &= \int_{T_0} |\nabla\psi_p^0|^2 d\xi d\eta \\ \int_{T_j} \nabla\psi_p^j \cdot \nabla\psi_{q_j}^j d\xi d\eta &= \int_{T_0} \nabla\psi_p^0 \cdot \nabla\psi_{q_0}^0 d\xi d\eta \\ \int_{T_j} \nabla\psi_p^j \cdot \nabla\psi_{q_{j+1}}^j d\xi d\eta &= \int_{T_0} \nabla\psi_p^0 \cdot \nabla\psi_{q_1}^0 d\xi d\eta\end{aligned}$$

for $0 \le j \le 5$, and evaluate each of the three integrals.

d) The variational formulation of (1) gives a bilinear form $a(u, v),\ u, v \in S$. Evaluate $a(\phi_p, \phi_p)$ and $a(\phi_p, \phi_{q_j})$, where $\phi_p,\ \phi_{q_j}$ are pyramid functions centered at p and q_j.

Exercise 3. Consider the hyperbolic problem

$$u_t + au_x = bu.$$

Lax-Wendroff's scheme for this equation is

$$U_m^{m+1} = (1 + bk + \frac{1}{2}(bk)^2)U_m^n - \frac{ap}{2}(1 + bk)(U_{m+1} - U_{m-1}^n) + \frac{(ap)^2}{2}\delta_x^2 U_m^n$$

where $p = k/h$ and k and h are stepsizes in time and space.

a) Show how this scheme can be derived, using for example that $\partial_t^2 u = a^2\partial_x^2 u - 2ab\partial_x u + b^2 u$.

b) Derive an expression for the local truncation error τ_m^n, taking into account the possibilities of having $a = 0$ or $b = 0$. You should at least have an expression of the form $\tau_m^n = \mathcal{O}(\ldots)$, for the leading terms.

c) Put $\zeta = bk,\ r = ap$. Show that the above scheme is von Neumann stable for all $\zeta,\ r$ such that

$$(1 + \zeta + \zeta^2/2)^2 + 4qr^2(\zeta + \zeta^2/2) + 4r^2q^2(r^2 - (1 + \zeta)^2) \le 1, \quad \text{for all } 0 \le q \le 1.$$

Hint. The parameter q emerges as $q = \sin^2\frac{\beta h}{2}$ with usual notation for von Neumann stability.

d) Show that the neccessary conditions for von Neumann stability is

$$-2 \le \zeta \le 0, \quad r^2 \le 1 + \frac{1}{2}\zeta + \frac{1}{4}\zeta^2.$$