Hyperbolic equations:

PDE: $a u_{xx} + 2b u_{xy} + c u_{yy} = f(x,y,u,u_x,u_y)$

This is hyperbolic if $b^2 - ac > 0$

Hyperbolic problems are time-dependent, initial/boundary value problems.

Classical example: Wave equation.

$$u_{tt} = c^2 u_{xx}.$$

Given the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, on $\mathbb{R}$
the exact solution is given by

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

(d'Alambert's solution.)

- The transport equation: (a is the velocity)

$$u_t + a u_x = 0 \qquad \Rightarrow \qquad u(x,t) = f(x-at)$$

more general:

$$u_t + a(x,t,u)u_x = 0$$

Burgers
$u_t + u \cdot u_x = 0$

- System of equations:

$$\vec{u}_t + A\vec{u}_x = 0,$$

Hyperbolic if A has a full set of real eigenvalues. (can be split into a set of transport equations.)

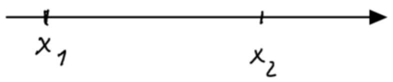
- Conservation form:

$$\vec{u}_t + (\vec{f}(\vec{u}))_x = 0, \qquad \vec{u} = [u_1, \cdots, u_n]^T$$

Example: Let $u = \rho$: The density of something.

$$\rho_t + (f(\rho))_x = 0$$

The total mass of s.t. is $\int_{x_1}^{x_2} \rho(x,t)\,dx$.



$$\int_{x_1}^{x_2} \rho_t\,dx + f(\rho)\Big|_{x_1}^{x_2} = 0$$

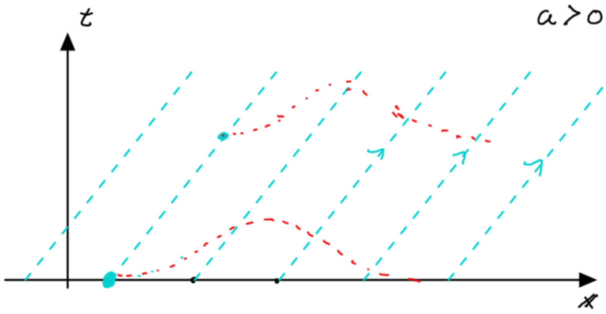
$$\Rightarrow \frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = f(\rho)\Big|_{x_1} - f(\rho)\Big|_{x_2}$$

Characteristics:

- $u_t + a u_x = 0$

  $u(x,0) = f(x)$

  $u(x,t) = f(x - at)$



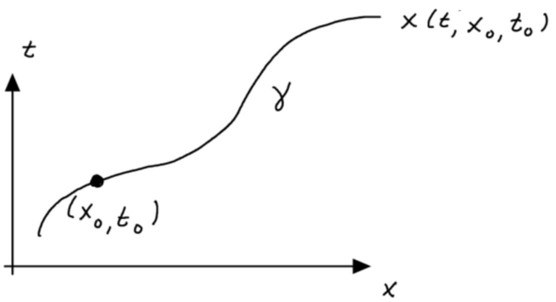
The characteristic is the straight line $x - at = \text{const.}$ and in our case, the solution is constant along those.

More general:

- $u_t + a(x,t) u_x = b(x,t)$

  Let $x(t; x_0, t_0)$ be the solution of the ODE

  $$\frac{dx}{dt} = a(x,t), \quad x(t_0) = x_0.$$



Let $v(t) = u(x(t, x_0, t_0), t)$ be the solution u along γ. Then

$$\frac{dv}{dt} = u_t + u_x \cdot \frac{dx}{dt}$$

$$= u_t + a(x,t) u_x = b(x(t; x_0, t_0), t) = \tilde{b}(t)$$

$$v(t) = v(t_0) + \int_{t_0}^{t} \tilde{b}(s)\,ds$$

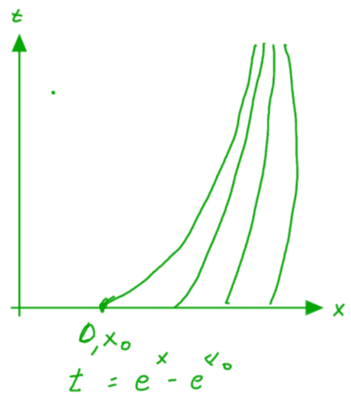
$$u(x(t; x_0, t_0), t) = u(x_0, t_0) + \int_{t_0}^{t} \tilde{b}(s)\,ds.$$

Example:

$$u_t + e^{-x} u_x = 0, \quad u(x,0) = f(x)$$

$$\gamma: \quad \frac{dx}{dt} = e^{-x}, \quad \Rightarrow e^{x} - e^{x_0} = t - t_0$$

$$e^{x} dx = dt$$
$$e^{x} - e^{x_0} = t - t_0$$



The characteristics is crucial when setting up boundary conditions:

$$u_t + a u_x = 0 \quad \text{on } 0 < x < 1, \quad u(x,0) = f(x)$$

If $a > 0$: $u(0,t) = \varphi_0(t)$ (No BC on $x = 1$)

$a < 0$: $u(1,t) = \varphi_1(t)$

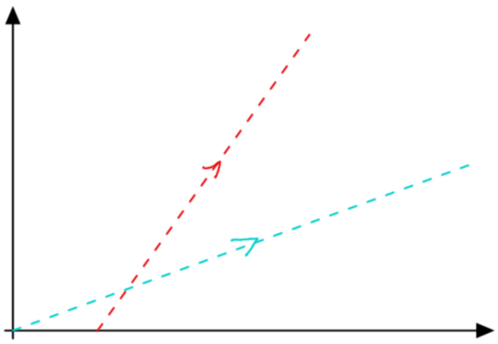
What about systems: $\lambda = a \pm b$

Example: $\vec{u}_t = \begin{bmatrix} a & b \\ b & a \end{bmatrix}^T \vec{u}$ on $0 \le x \le 1$, $t \ge 0$

This system can be rewritten $v_1 = u_1 + u_2$, $v_2 = u_1 - u_2$

$$v_{1,t} + (a+b)v_{1,x} = 0$$
$$v_{2,t} + (a-b)v_{2,x} = 0$$

$b<a$

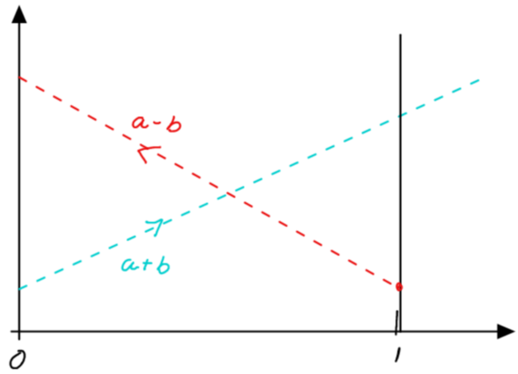


$v_1(0,t) = g_0(t)$
$v_2(0,t) = g_1(t)$

$\Downarrow$

$u_1(0,t) = \varphi_0(t)$
$u_2(0,t) = \varphi_1(t)$ can be chosen freely

$b>a$



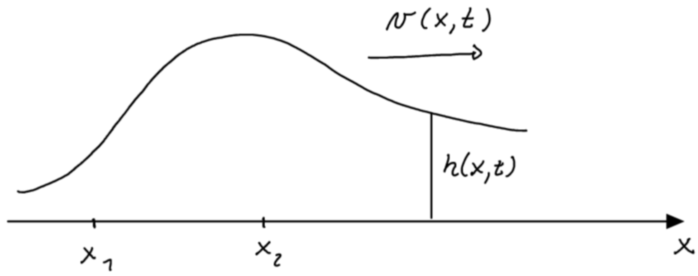
$v_1(0,t) = g_0(t)$
$v_2(1,t) = g_1(t)$

BC on both sides. And these has to satisfy

$u_1(0,t) + u_2(0,t) = g_0(t)$
$u_1(1,t) - u_2(1,t) = g_1(t)$

Shallow water equation:

- Fluid in a channel.
  $\bar{\rho}$: Density, constant.
  $h(x,t)$: height
  $v(x,t)$: velocity
  $p(x,t)$: pressure



Mass conservation:

Total mass in $[x_1, x_2] = \int_{x_1}^{x_2} \rho \cdot h(x,t)\,dx$

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, h(x,t)\,dx = v \cdot \rho \cdot h \Big|_{x=x_1} - v\rho h \Big|_{x=x_2} = -\int_{x_1}^{x_2} (v\rho h)_x\,dx$$

$$\Rightarrow \quad h_t + (vh)_x = 0$$

Momentum: (mass $\cdot$ velocity): $\int_{x_1}^{x_2} \rho \cdot h \cdot v\,dx = \rho \cdot hv \cdot v \Big|_{x_2}^{x_1}$

$$(\cancel{\rho} h v)_t + (\cancel{\rho} h v^2 + p)_x = 0, \quad p = \frac{1}{2}\cancel{\rho}\, \underset{\text{gravitation}}{g}\, h^2 \qquad p = \int_0^h \rho g y\,dy = \frac{1}{2}\rho g h^2$$

$$(hv)_t + (hv^2 + \frac{1}{2}gh^2)_x = 0$$

$$h_t \cdot v + v_t \cdot h = -(vh)_x \cdot h + v_t \cdot h$$

$$(hv)_x v + h \cdot v \cdot v_x + g \cdot h \cdot h_x$$

$$\Rightarrow \quad v_t + v \cdot v_x + g \cdot h_x = 0 \quad \Rightarrow \quad v_t + (\frac{1}{2}v^2 + gh)_x = 0$$

$$h_t - (v \cdot h)_x = 0$$

Let $z = g \cdot h \quad \Rightarrow \quad z_t + (v \cdot z)_x = 0$

$$v_t + (\frac{1}{2}v^2 + z)_x = 0$$

Numerical schemes

Model equation:

$$u_t + a u_x = 0, \quad a > 0 \quad \text{on } 0 \le x \le 1$$

IV: $u(x,0) = f(x)$
BV: $u(0,t) = \varphi(t)$

Exact solution: $u(x,t) = \begin{cases} f(x-at) & x \ge at \\ \varphi_0(t - \frac{1}{a}x) & x < at \end{cases}$

NB! This is an I/BVP, time dependent, so the theory from parabolic equations apply:

$$\boxed{\begin{matrix}\text{consistency} \\ \text{stability}\end{matrix} \iff \text{convergence.}}$$

Forward in time and space:

$$\frac{1}{k}\Delta_t U_m^n + \frac{1}{h} a \Delta_x U_m^n = 0$$

$$U_m^{n+1} - U_m^n + a \cdot \frac{k}{h}(U_{m+1}^n - U_m^n) = 0$$

$$\Rightarrow U_m^{n+1} = U_m^n - p(U_{m+1}^n - U_m^n) \qquad \text{consistent.}$$

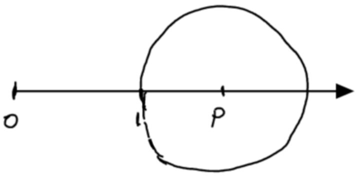
Courant number

$$p = a \cdot \frac{k}{h}$$

Von Neumann stability: $U_m^n = g^n e^{i\beta x_m}$,

$$g^{n+1} e^{i\beta x_m} = g^n e^{i\beta x_m} - p(g^n e^{i\beta(x_m+h)} - g^n e^{i\beta x_m})$$

$$g = 1 - p \cdot (e^{i\beta h} - 1) = (1+p) - p \cdot e^{i\beta h}$$



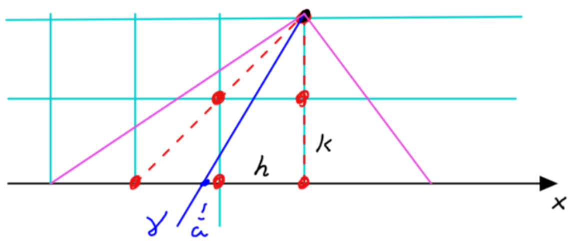
Stable only if $-1 \le p \le 0$ which is impossible if $a > 0$.

What if $U_m^{n+1} = U_m^n - a\frac{k}{h}(U_m^n - U_{m-1}^n)$

$$g = 1 - p(1 - e^{i\beta h})$$

$$|g| \le 1 \quad \text{if} \quad 0 < p < 1$$

- How does this fit with what we know about our solution



Region of dependency

The characteristic has to stay in the region of dependency!

$$\frac{k}{h} \le \frac{1}{a} \Rightarrow 0 \le a\frac{k}{h} \le 1$$