**Final Exam solution**

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**1** $A_{a,b,c}$ is symmetric if $(A_{a,b,c})_{ij} = (A_{a,b,c})_{ji} \iff b = c$
$A_{a,b,c}$ is positive definite if all eigenvalues $\lambda_i > 0$

$$\iff 0 < \lambda_2 = 2 - \sqrt{2b^2} = 2 - \sqrt{2}|b|$$
$$\iff |b| < \sqrt{2}.$$

Hence $A_{a,b,c}$ is symmetric positive definite (SPD) iff

$$c = b \text{ and } |b| < \sqrt{2}.$$

Let $A = D - R$, $D = 2I$, $R = 2I - A$. The Jacobi iteration for $A\vec{x} = \vec{b}$ is then:

$$(*) \qquad D\vec{x}^{n+1} = R\vec{x}^n + \vec{b} \iff \vec{x}^{n+1} = D^{-1}R\vec{x}^n + D^{-1}\vec{b}$$

where $B = D^{-1}R = \frac{1}{2}R = \frac{1}{2}(2I - A) = \text{tridiag}\left\{-\frac{b}{2}, 0, -\frac{b}{2}\right\}$. By the formula for the eigenvalus given in the problem:

$$\lambda_1 = 0, \quad \lambda_2 = -\sqrt{2\frac{b^2}{4}} = -\frac{|b|}{\sqrt{2}}, \quad \lambda_3 = +\sqrt{2\frac{b^2}{4}} = \frac{|b|}{\sqrt{2}}$$

Hence

$$\rho(B) = \max|\lambda_i| = \frac{|b|}{\sqrt{2}} < 1 \quad \text{(since } A \text{ is SPD)}$$

and $(*)$ is convergent.

**2** **a)** Backward in time FDM:

$$\frac{\nabla_k U_m^{n+1}}{k} - \frac{\delta_x^2 U_m^{n+1}}{h^2} + x_m \frac{\delta_{2h} U_m^{n+1}}{2h} = 1$$
$$\iff U_m^{n+1} - U_m^n - \frac{k}{h^2}(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) + x_m \frac{k}{2h}(U_{m+1}^{n+1} - U_{m-1}^{n+1}) = 1$$
$$\iff (1+2r)U_m^{n+1} - r\left(1 - \frac{x_m}{2}h\right)U_{m+1}^{n+1} - r\left(1 + \frac{x_m}{2}h\right)U_{m-1}^{n+1} = U_m^n + 1$$

where $r = \frac{k}{h^2}$. The boundary conditions are $U_0^n = 1$ and $U_m^n = e$, and the resulting linear system becomes:

$$A_h\vec{U}^{n+1} = \vec{U}^n + \vec{F}^n, \qquad \vec{U}^0 = \vec{G},$$

where $A_h$ is the tridiagonal matrix given by

$$A_h = \begin{bmatrix} 1+2r & -r\left(1-\frac{x_1}{2}h\right) & 0 & \dots & 0 \\ -r\left(1+\frac{x_2}{2}h\right) & 1+2r & -r\left(1-\frac{x_2}{2}h\right) & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & -r\left(1+\frac{x_{M-2}}{2}h\right) & 1+2r & -r\left(1-\frac{x_{M-2}}{2}h\right) \\ 0 & \dots & 0 & -r\left(1+\frac{x_{M-1}}{2}h\right) & 1+2r \end{bmatrix},$$

$$\vec{F}^n = k\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -r\left(1+\frac{x_1}{2}h\right)\cdot 1 \\ 0 \\ \vdots \\ 0 \\ -r\left(1-\frac{x_{M-1}}{2}h\right)\cdot e \end{bmatrix}, \quad \text{and} \quad \vec{G} = \begin{bmatrix} e^{x_1} \\ \vdots \\ e^{x_m} \\ \vdots \\ e^{x_{M-1}} \end{bmatrix}.$$

**b)** Backward in time FDM:

$$\frac{\Delta_k U_m^n}{k} - \frac{\delta_x^2 U_m^n}{h^2} + x_m \frac{\delta_{2h} U_m^{n+1}}{2h} = 1$$

$$\iff U_m^{n+1} - U_m^n - \frac{k}{h^2}(U_{m+1}^n - 2U_m^n + U_{m-1}^n) + x_m \frac{k}{2h}(U_{m+1}^n - U_{m-1}^n) = 1$$

$$\iff U_m^{n+1} - \underbrace{\left(1 - 2\frac{k}{h^2}\right)}_{\alpha_{m,m}^n} U_m^n - \underbrace{\frac{k}{h^2}\left(1 - \frac{x_m}{2}h\right)}_{\alpha_{m,m+1}^n} U_{m+1}^n - \underbrace{\frac{k}{h^2}\left(1 + \frac{x_m}{2}h\right)}_{\alpha_{m,m-1}^n} U_{m-1}^n = 1.$$

Hence with $\alpha_{m,m}^n$, $\alpha_{m,m\pm1}^1$ defined above and $\alpha_m^{n+1} = 1$,

$$\alpha_m^{n+1} U_m^{n+1} - \sum \alpha_{m,l}^n U_l^n = F_m^n = 1.$$

The scheme is monotone if

$$\alpha_m^{n+1} > 0 \quad \text{(ok!)},$$

$$\alpha_{m,m}^n \geq 0 \iff 1 - 2\frac{k}{h^2} \geq 0 \iff \frac{k}{h^2} \leq \frac{1}{2} \quad \text{(condition!)},$$

$$\alpha_{m,m+1}^n \geq 0 \iff 1 - \frac{x_m}{2}h \geq 0 \iff h \leq \frac{2}{x_m} \underset{x_m \leq 1}{\iff} h \leq 2 \quad \text{(ok on } [0,1]),$$

$$\alpha_{m,m-1}^n \geq 0 \quad \text{(ok!)},$$

$$\alpha_m^{n+1} \geq \sum_l |\alpha_l^n|, \quad \text{ok since } \alpha_m^{n+1} = 1 = \sum_l \alpha_l^n.$$

CFL condition: $k \leq \frac{1}{2}h^2$.

**3** **a)** Local truncation error:

$$\tau_m = -(L - L_h)u(x_m) = -\left(\partial_x^2 - x_m\partial_x - 1 - \left(\frac{\delta_x}{h^2} - x_m\frac{\nabla_h}{h} - 1\right)\right)u(x_m)$$

$$= \underbrace{\left(\frac{\delta_h^2}{h^2} - \partial_x^2\right)u(x_m)}_{=I_1} - x_m \underbrace{\left(\frac{\nabla_h}{h} - \partial_x\right)u(x_m)}_{=I_2}$$

Taylor:

$$u_{m\pm1} = u_m \pm h(u')_m + \frac{1}{2}h^2(u'')_m \pm \frac{1}{6}h^3(u''')_m + \frac{1}{4!}h^4u^{(4)}(\xi_m^\pm),$$

$$I_1 = \frac{1}{h^2}(u_{m+1} - 2u_m + u_{m-1}) - u''_m$$

$$= 0\cdot u_m + 0\cdot(u')_m + \cancel{\left(\frac{1}{2}+\frac{1}{2}\right)\frac{h^2}{h^2}(u'')_m} + 0\cdot(u''')_m + \frac{1}{24}h^2 \underbrace{\left(u^{(4)}(\xi_m^+) + u^{(4)}(\xi_m^-)\right)}_{=2u^{(4)}(\xi_m)\text{ by mean val. thm.}} - \cancel{u''_m}$$

$$= \frac{1}{12}h^2u^{(4)}(\xi_m),$$

$$I_2 = \frac{1}{h}(u_m - u_{m-1}) - (u')_m = \frac{1}{h}\left(\cancel{u_m - u_m} - \cancel{h(u')_m} + \frac{1}{2}h^2u''(\eta_m)\right) - \cancel{(u')_m} = \frac{1}{2}hu''(\eta_m),$$

where $\xi_m \in (x_{m-1}, x_{m+1})$ and $\eta_m \in (x_{m-1}, x_m)$. Hence

$$\tau_m = I_1 - x_mI_2 = \frac{1}{12}h^2u^{(4)}(\xi_m) - x_m\frac{1}{2}hu''(\eta_m).$$

**b)** The scheme is monotone so the discrete maximum principle (DMP) holds (the boundary connectivity condition is trivially satisfied here):

$$-L_h\tilde{V}_m \le 0, \quad m = 1,\dots,M-1 \Rightarrow \max_{m=1,\dots,M-1}\tilde{V}_m \le \max\{\tilde{V}_0, \tilde{V}_m, 0\}$$

Define $W_m = V_m - \|f\|_\infty \cdot 1$ and note that

$$-L_hW_m = -L_hV_m + \|f\|_\infty(-L_h1) \le f_m - \|f\|_\infty \le 0, \quad m = 1,\dots,M-1,$$

and $W_0 = V_0 - \|f\|_\infty \le 0$, $W_M = V_M - \|f\|_\infty \le 0$. By DMP

$$\max_m W_m \le \max\{0, W_0, W_M\} \le 0$$
$$\Downarrow \quad W_m = V_m - \|f\|_\infty$$
$$\max_m V_m \le \|f\|_\infty$$

Since $-L_h(-V_m) = -f_m$, we also get $\max_m(-V_m) \le \|f\|_\infty$. Hence $\|V\|_\infty \le \|f\|_\infty$.

**c)** First note that by a),

$$|\tau_m| \le \frac{1}{12}h^2\|u^{(4)}\|_\infty + \frac{1}{2}h\|u''\|_\infty, \qquad m = 1,\dots,M-1.$$

Let the error $e_m := u(x_m) - U_m$. We find the error equation and use stability and the truncation error estimate to conclude:

$$\begin{cases} -L_he_m = -L_hu_m - (-L_hU_m) = -(Lu_m + \tau_m) + L_hU_m = 1 - \tau_m - 1 = \tau_m \\ e_0 = 0 = e_m \end{cases}$$
$$\Downarrow b)$$
$$\max|e_m| \le \|-\tau\|_\infty$$
$$\Downarrow a)$$
$$\max|u(x_m) - U_m| \le \frac{1}{2}h\|u''\|_\infty + \frac{1}{12}h^2\|u^{(4)}\|_\infty.$$

**4** **a)** Multiply (4) by a smooth function $v$ with $v(\frac{\pi}{6}) = 0 = v(\frac{\pi}{2})$, integrate, and then integrate by parts:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \; v \,\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} -\partial_x(\sin x \; u_x) v \,\mathrm{d}x = \underbrace{[-\sin x \; u_x v]_{\frac{\pi}{6}}^{\frac{\pi}{2}}}_{\to 0} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (-\sin x \; u_x) v_x \,\mathrm{d}x$$

Hence $a(u, v) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \; u_x v_x \,\mathrm{d}x$ and $F(v) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \; v \,\mathrm{d}x$.

Note that $a, F$ are well defined for $u, v \in X = H^1\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. To have $v(0) = 0 = v(1)$, take $v \in V := H_0^1\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ ($u \notin V$ because of the boundary conditions).

$a(\cdot, \cdot)$ is bilinear: For $u_1, u_2, v \in X = H^1(\frac{\pi}{6}, \frac{\pi}{2})$, $c_1, c_2 \in \mathbb{R}$,

$$a(c_1 u_1 + c_2 u_2, v) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x (c_1 u_{1,x} + c_2 u_{2,x}) \,\mathrm{d}x = c_1 a(u_1, v) + c_2 a(u_2, v),$$

and $a$ is linear in the first argument. Since $a(u, v) = a(v, u)$ (symmetric), this also implies linearity of the second argument.

$a(\cdot, \cdot)$ is a continuous form: For $u, v \in X = H^1(\frac{\pi}{6}, \frac{\pi}{2})$,

$$|a(u, v)| \leq \|\sin x\|_{L^\infty} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} |u_x||v_x| \,\mathrm{d}x \leq 1 \cdot \|u_x\|_{L^2} \|v_x\|_{L^2} \leq \|u\|_X \|v_x\|_X,$$

since $\|v\|_X^2 = \|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 \geq \|v_x\|_{L^2}^2$. The second inequality follows from the Cauchy-Schwartz inequality.

**b)** Finite element method: Find $u_h = \sum_{i=0}^{3} U_i \varphi_i \in X_h^1$ s.t.

$$(*) \qquad u_h(x_0) = 1, \quad u_h(x_3) = 0,$$
$$(**) \qquad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h := X_h^1 \cap H_0^1.$$

where $x_0 = \frac{\pi}{6}, \ldots, x_3 = \frac{\pi}{2}$ and $\varphi_j(x_i) = \delta_{ij}$. By $(*)$:

$$1 = u_h(x_0) = \sum_{i=0}^{3} U_i \varphi_i(x_0) = U_0 \cdot 1 + 0,$$
$$0 = u_h(x_3) = \sum_{i=0}^{3} U_i \varphi_i(x_3) = 0 + U_3 \cdot 1.$$

Since $v_h \in \operatorname{span}\{\varphi_0, \ldots, \varphi_3\}$,

$$(**) \iff a(u_h, \varphi_j) = F(\varphi_j), \quad j = 0, \ldots, 3,$$
$$\iff \sum_{i=0}^{3} a(\varphi_i, \varphi_j) U_i = F(\varphi_j), \quad j = 0, \ldots, 3.$$

Combining:

$$\begin{cases} U_0 = 1 \\ \sum_{i=0}^{3} a(\varphi_i, \varphi_1) U_i = F(\varphi_1) \\ \sum_{i=0}^{3} a(\varphi_i, \varphi_2) U_i = F(\varphi_2) \\ U_3 = 0 \end{cases}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ a(\varphi_0,\varphi_1) & a(\varphi_1,\varphi_1) & a(\varphi_2,\varphi_1) & a(\varphi_3,\varphi_1) \\ a(\varphi_0,\varphi_2) & a(\varphi_1,\varphi_2) & a(\varphi_2,\varphi_2) & a(\varphi_3,\varphi_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=A} \begin{bmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 \\ F(\varphi_1) \\ F(\varphi_2) \\ 0 \end{bmatrix}$$

Observations:

(i) $c := a(\varphi_1, \varphi_2) = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, \partial\varphi_1 \partial\varphi_2 \, \mathrm{d}x = a(\varphi_2, \varphi_1)$,

(ii) $\operatorname{supp} \varphi_j = K_j \cup K_{j+1} \Rightarrow \partial\varphi_0\partial\varphi_2 = 0 = \partial\varphi_3\partial\varphi_1$ in $\left[\frac{\pi}{6}, \frac{\pi}{2}\right] \Rightarrow a(\varphi_0, \varphi_2) = 0 = a(\varphi_3, \varphi_1)$.

Hence

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & * & c & 0 \\ 0 & c & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To compute $c$, we need $\partial\varphi_j$, $j = 1, 2$. Let $h_j = x_j - x_{j-1}$ be the length of $K_j$. Since $\varphi_j$ is continuous and piecewise linear with $\varphi_j(x_k) = \delta_{jk}$, it follows that

$$\partial\varphi_j(x) = \begin{cases} \frac{1}{h}_j, & x \in K_j, \\ -\frac{1}{h}_{j+1}, & x \in K_{j+1}, \\ 0, & \text{otherwise}, \end{cases} \implies \varphi_1(x)\varphi_2(x) = \begin{cases} -\frac{1}{h}_2 \frac{1}{h}_2, & x \in K_2, \\ 0, & \text{otherwise}. \end{cases}$$

Then we compute

$$\begin{aligned} c = a(\varphi_1, \varphi_2) &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, \partial\varphi_1 \partial\varphi_2 \, \mathrm{d}x = \int_{K_2} \sin x \left( -\frac{1}{h_2^2} \right) \mathrm{d}x \\ &= \frac{1}{\left(\frac{\pi}{3} - \frac{\pi}{4}\right)^2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (-\sin x) \, \mathrm{d}x = \frac{12^2}{\pi^2} [\cos(x)]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\ &= \frac{144}{\pi^2} \left( \frac{1}{2} - \frac{\sqrt{2}}{2} \right) = -\frac{72}{\pi^2}(\sqrt{2} - 1). \end{aligned}$$