TMA4212 Numerical solution of differential equations by difference methods - Spring 2022.
https://wiki.math.ntnu.no/tma4212/2022v/start

Lecture 1 (Tue)

A. Recap PDEs

a) Linear PDEs in $\mathbb{R}^2$ [Mat 4K]

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + g u = f \quad \text{in } \Omega$$

where

$\Omega \subset \mathbb{R}^2$, domain

$u = u(x,y) : \Omega \to \mathbb{R}$, sol'n

$a, \dots, g : \Omega \to \mathbb{R}$, coeff's / data

$u_x = \partial_x u = \frac{\partial u}{\partial x}$, $u_{xy} = \partial_x \partial_y u$ etc.

Classification:

		Typical ex.	
$b^2 - ac < 0$	elliptic	$u_{xx} + u_{yy} = f$	(Poisson)
$b^2 - ac = 0$	parabolic	$u_t \underset{y=t}{=} u_{xx}$	(Heat eq'n)
$b^2 - ac > 0$	hyperbolic	$u_{tt} \underset{y=t}{=} u_{xx}$	(wave eq'n)

b) Bnd'ry/init. cond'ns (B./I.C.)

I.C. : u given at $t = 0$

Dirichlet B.C. : u given at $\partial\Omega$

Neumann B.C. : $\frac{\partial u}{\partial n}$ — " —

Mixed, Robin, ...

Need B./I.C. to have unique sol'n of PDE.

c) Nonlinear PDEs

$u_t + uu_x = -p_x + \nu u_{xx}$ (fluid mech.)

$u_t = \partial_x(u\partial_x u)$ (porous medium)

$u_x^2 + u_y^2 = f$ (Eikonal eq'n)

...

d) PDE models

- Model phys. phenomena
- Should be well-posed (as is nature):
  sol'ns exist, are unique, and dep. cont. on data (~ stability!!)
- Are very different, no unifying theory

Only the simplest ones can be solved analytically!

B. About the course

Intro. to num. sol'n of PDEs:

- Konstruere num. met.
- Løse disse met. (implementere og teste)
- Teoretisk analyse (løsbare, stabile, konv.)
- Verifisere teori (num. forfining, feil, rater)

Metoder:

Endelig differansemetoder (FDM) – hovedfokus (enkle!)
Endelig elementmetoder (FEM) – moderne / utbredt

Problemer: Lin. B/I VP (bnd'ry/init. value prob's)

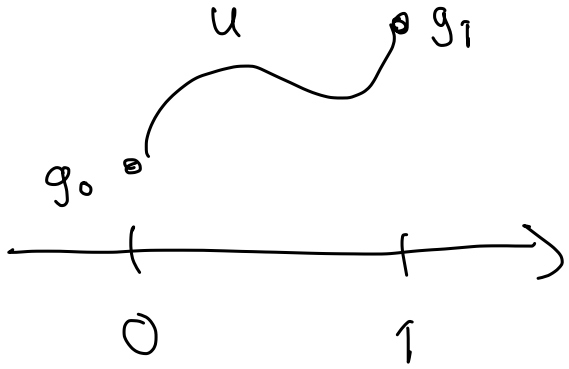
1d:	ODE's
1d + tid	Par. PDE'er
2d	El. PDE'er
1d + time	Hyp. PDE'er

FEM for 1D/2D ell. PDE'er

Part 1: FDM for 1d BVP'er

C. FDM for 1d Poisson likn.

$$(P)\quad \begin{cases} -u_{xx} = f(x) & \text{in } (0,1) \quad \text{domene} \quad (PDE) \\ u(0)=g_0,\ u(1)=g_1 & (BC) \end{cases}$$



where $f : [0,1] \to \mathbb{R}$; $g_0, g_1 \in \mathbb{R}$

Rem 1: (a) $f \in C^k([0,1]) \Longrightarrow$ (P) har entydig løsn $u \in C^{k+2}([0,1])$

(b) Hvis $g_0 = 0 = g_1$, da er $u(x) = \int_0^1 G(x,y) f(y)\,dy$, $G(x,y) = \begin{cases} (1-y)x, & 0 \le x \le y \le 1 \\ y(x-1), & 0 \le y \le x \le 1 \end{cases}$

Discretisation:

1) Def. a grid G_h on $[0,1]$:

$$x_m = m \cdot h, \quad m = 0, \dots, M, \quad h = \frac{1}{M}$$

M: no of intervals; h: step size

2) Replace derivatives in (P) by differences:

$$u_{xx}(x_m) \simeq \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} =: \frac{1}{h^2}\,\delta_h^2\, u_m$$

where $u_m = u(x_m)$.

3) Replace u by discrete approx. $U : \mathcal{G}_h \to \mathbb{R}$:

$$U_m \simeq u(x_m), \quad m = 0, \dots, M$$

Scheme:

$$(P_h) \begin{cases} \frac{1}{h^2}(-U_{m+1} + 2U_m - U_{m-1}) = f_m, & m = 1, \dots, M-1 \\ U_0 = g_0, \quad U_M = g_1 \end{cases}$$

or (mult. by h^2)

$$(\tilde{P}_h) \quad A_h \vec{U} = \vec{F}, \quad A_h = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} f_1 + \frac{g_0}{h^2} \\ f_2 \\ \vdots \\ f_{M-1} + \frac{g_1}{h^2} \end{bmatrix}$$

$(M-1)(M-1)$ $\qquad$ $(M-1)$

where $\vec{U} = (U_1, \dots, U_{M-1})$

PLS CHK !

D. Error-analysis

Error: $e_{m,h} = u_m - U_m$; $[e_0 = 0 = e_M]$

We want

Convergence: $\|\vec{e}_h\| \longrightarrow 0$ as $h \to 0$

where $\|\cdot\|$ is a norm and $\vec{e}_h = (e_{h,1}, \dots, e_{h,M-1})$

Error eq'n: ("subtract (P) and (P_h)")

$$(*)\quad A_h \vec{e}_h = A_h \vec{u} - A_h \vec{U} \overset{(\tilde{P}_h)}{=} A_h \vec{u} - \vec{F} \overset{(P)}{=} \underbrace{A_h \vec{u} - \left(\frac{1}{h^2} \begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix} - \vec{u}_{xx} \right)}_{= -\frac{1}{h^2}\delta_h^2 \vec{u}} =: \vec{\tau}_h$$

$\vec{f} + \frac{1}{h^2}\begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix}$

where truncation error

$$\tau_{h,m} := u_{xx}(x_m) - \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2}, \quad m = 1, \dots, M-1$$

Prop. 1: If (P) is well-posed (in our case it is) and (P_h) is

(C) consistent: $\|\vec{\tau}_h\| \longrightarrow 0$ as $h \to 0$

(S) stable: For any $\vec{g}$ and sol'n $\vec{V}$ of $A_h \vec{V} = \vec{g}$ and $V_0 = 0 = V_M$,

$$\|\vec{V}\| \le C \|\vec{g}\|, \quad C \text{ indep. of } h \text{ and } \vec{g}.$$

then (P_h) is conv. and $\|\vec{u} - \vec{u}\| \le C \|\vec{\tau}_h\|$.

Rem. 2: If $\|A_h^{-1}\| \le C$ <u>indep. of h</u>, then (S) holds:

$$\|\vec{V}\| \overset{(P_h)}{=} \|A_h^{-1}\vec{g}\| \le \|A_h^{-1}\| \cdot \|\vec{g}\| \le C\|\vec{g}\|$$

Pf. Prop. 1:

Obs: $\vec{V} = \vec{e}_h$ solves $A_h\vec{V} = \vec{g}$ w. $\vec{g} = \vec{\tau}_h$, $V_0 = 0 = V_M$ (see (*))

$$\Rightarrow \|\vec{e}_h\| \overset{(S)}{\le} C\|\vec{\tau}_h\| \overset{(C)}{\longrightarrow}_{h\to 0} 0$$

□

E. Convergence in l^∞ / L^∞

Take $\|\vec{x}\| = \|\vec{x}\|_\infty = \max_{i=1,\dots,M-1} |x_i|$

Consistency (C):

$$u_{m\pm1} = u(x_m \pm h) \overset{\text{Taylor}}{=} u_m \pm h(u_x)_m + \frac{1}{2}h^2(u_{xx})_m \pm \frac{1}{3!}h^3(u^{(3)})_m + \frac{1}{4!}h^4 u^{(4)}(\xi_\pm)$$

$$\Downarrow$$

$$u_{m+1} - 2u_m + u_{m-1} \overset{\text{chk!}}{=} h^2(u_{xx})_m + \underbrace{\frac{1}{24}h^4\left(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)\right)}_{h^2\tau_{h,m}}$$

$\Downarrow \cdot \frac{1}{h^2}$

(T) $\|\vec{\tau}_h\|_\infty = \max_m |\tau_{h,m}| \le K h^2, \quad K = \frac{1}{12} \max_{x \in [0,1]} |u^{(4)}(x)|$

Stability (S):

Lem. 2: If $\vec{V}$ solves $A_h \vec{V} = \vec{g}$ and $V_0 = 0 = V_m$, then $\|\vec{V}\|_\infty \le \frac{1}{8} \|\vec{g}\|_\infty$.

Pf.: Use discr. max. pr. $\longrightarrow$ next time

Obs: $C = \frac{1}{8}$ in (S)

By Prop. 1:

$$\|\vec{u} - \vec{u}\| \le C \|\vec{\tau}_h\| = \frac{1}{8} \cdot \frac{1}{12} K \cdot h^2, \quad K = \max |u^{(4)}|$$

Rem. 3:

a) 2nd order meth.

b) $f \in C^2([0,1]) \underset{\text{Rem. 1}}{\Longrightarrow} K < \infty.$

Lecture 2 (Thu)

A. Recap

(P) $\quad -u_{xx} = f \quad \text{in} \quad (0,1); \qquad u(0) = g_0, \quad u(1) = g_1$

FDM on grid G_h: $x_m = m \cdot h, \quad m = 0, \dots, M, \quad h = \frac{1}{M}$:

(P_h) $\quad -\frac{1}{h^2} \delta_h^2 U_m = f_m, \quad m = 1, \dots, M-1; \qquad U_0 = g_0, \quad U_M = g_1$

$\Updownarrow$

where $\delta_h^2 U_m = U_{m+1} - 2U_m + U_{m-1}$

$(\tilde{P}_h)$ $\quad A_h \vec{U} = \vec{F}; \qquad A_h = \frac{1}{h^2} \text{tridiag}\{-1, 2, -1\}, \quad \vec{F} = [f_1 + \frac{g_0}{h^2}, f_2, \dots, f_{M-1} + \frac{g_1}{h^2}]$

Prop.: $\quad \|\vec{U} - \vec{u}\|_\infty \leq \frac{1}{96} h^2 \cdot \max_{[0,1]} |u^{(4)}(x)|$

Consistant + stable $\Rightarrow$ conv. and error bnd.

B. Stability in L^∞ / l^∞

Lem 2 (last time): If $\vec{V}$ solves $A_h \vec{V} = \vec{g}$ and $V_0 = 0 = V_M$, then $\|\vec{V}\|_\infty \le \frac{1}{8}\|\vec{g}\|_\infty$

We will use the discr. max. pr.:

Lem 3: $\alpha_m, \beta_m, \gamma_m > 0$, $\beta_m \ge \alpha_m + \gamma_m$, and $\{v_m\}_{m=0}^M$ s.t.

(1) $$-\alpha_m v_{m+1} + \beta_m v_m - \gamma_m v_{m-1} \le 0, \quad m = 1, \dots, M-1$$

Then $$\max_m v_m \le \max\{v_0, v_M, 0\}$$

Rem. 1: v has max. at bnd'ry (like a (sub)harmonic func'n)

Pf.: Assume by contradiction there is p, $1 \le p \le M-1$ s.t.

(2) $$v_p = \max_{m=0,\dots,M} v_m > K_0 \quad (\ge 0)$$

Then

(3) $$v_p \overset{(1)}{\le} \frac{1}{\beta_p}(\alpha_p v_{p-1} + \gamma_p v_{p+1}) \le \frac{\alpha_p + \gamma_p}{\beta_p} v_p \le 1 \cdot v_p$$

(i) $\alpha_p + \gamma_p < \beta_p \overset{(3)}{\Rightarrow} v_p < 1 \cdot v_p$, contradiction $\Rightarrow$ Lem 3 ok

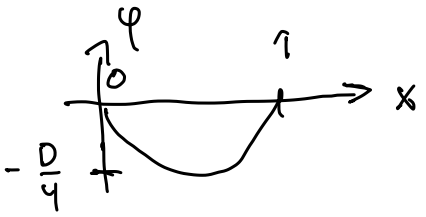
(ii) $\alpha_p + \gamma_p = \beta_p \overset{(3)}{\Rightarrow} v_p = \frac{1}{\beta_p}(\alpha_p v_{p-1} + \gamma_p v_{p+1}) \Rightarrow v_p = v_{p-1} = v_{p+1}$

Repeat arg. for $v_{p\pm1} \Rightarrow$ (i) or (ii): $v_p = v_{p\pm1} = v_{p\pm2}, \dots$

If (i) never happens, then $v_p = v_m = K_0 \ \forall m$, contradiction

$\Rightarrow$ Lem. 3 holds. □

Pf. of Lem. 2:



1) $\varphi(x) = Dx(x-1)$, $\quad D = \frac{1}{2}\max_m |g_m|$

$$\Rightarrow -\delta_h^2 \varphi_m = -\varphi_{m-1} + 2\varphi_m - \varphi_{m+1} \overset{\text{Chk. Taylor}}{=} -2Dh^2 \ ; \quad \varphi_0 = 0 = \varphi_M$$

2) $W_m = V_m + \varphi_m$

$$\Rightarrow W_0 = 0 = W_M, \quad -\delta_h^2 W_m \overset{\text{1)}}{=} g_m h^2 - 2Dh^2 = h^2(g_m - 2D) \le 0$$

$$\overset{\text{Lem. 3}}{\Rightarrow} \max_m W_m \le \max\{W_0, W_M, 0\} = 0$$

$$\Rightarrow \max_m V_m \overset{V \le W + |\varphi|}{\le} \max_m W_m + \max_m |\varphi_m| \le 0 + \frac{1}{4}D$$

3) $(u, g) \mapsto (-u, -g)$ in 2) $\Rightarrow \max(-V_m) \le \frac{1}{4}D$

4) 2) + 3) $\Rightarrow \max_m |V_m| \le \frac{1}{4}D = \frac{1}{8}\|\vec{g}\|_\infty$ □

Rem. 2:

a) Monotone meth. = meth. satisfying discr. max. pr.

b) Meth. w. pos. coeff. :

$$\beta_m u_m - \sum_{i=p}^{N} \gamma_{m,i} u_{m+i} = 0 \; ; \quad \gamma_{m,i} \geq 0, \; \beta_m \geq \sum \gamma_{m,i} \; (>0)$$

are monotone [pf. is the same as for Lem. 3].

C. Rem. on num. testing

Obs. 1: Any C^2- funct'n h solves (P) if we take $f(x) = -h_{xx}(x)$ and $g_0 = h(0)$, $g_1 = h(1)$!

Verify code and theory on test-ex. w. known sol'ns :

- choose a sol'n (h) (not too trivial)
- find f and B.C. as Obs. 1

Chk. rate/error bnd. by refining the grid :

Assume

$$E(h) = \max|e_m| = C h^p \quad \Rightarrow \quad \log E(h) = p \log h + \log C \quad [\text{lin. in } p]$$

To find the rate p:

- choose seq'nce of step sizes $h_1, \dots, h_N$

  (e.g. $h_i = h_0 \cdot 2^{-i}$)

- compute num. $E(h_1), \dots, E(h_N)$

- find p by lin. regression (polyfit in python) on $\{(\log h_i, \log E(h_i)\}_{i=1,\dots,N}$

- make a log log plot of $\{(h_i, E(h_i))\}$ (p will be the slope)

How does the result compare to theory:

$$E(h) \leq \frac{1}{96} \max_{[0,1]} |u^{(4)}(x)| \cdot h^2 \quad ??$$

D. On difference approx.

$u(x)$ func'n on $\mathbb{R}$

Grid $x_m = x_0 + m \cdot h$, $m \in \mathbb{N}_0$, $x_0 \in \mathbb{R}$, $u_m = u(x_m)$

Difference formula (Fd):

$$L_h u_m = \sum_{i=p}^{q} a_i u_{m+i}, \quad a_i \in \mathbb{R}; \quad p, q \in \mathbb{N}_0$$

Difference formula (1d):

$$L_h u_m = \sum_{l=p}^{q} a_l u_{m+l}, \quad a_l \in \mathbb{R}; \quad p, q \in \mathbb{N}_0$$

Obs:

$$u_{m+l} \overset{\text{Taylor}}{=} \sum_{k=0}^{v} \frac{(lh)^k}{k!} u_m^{(k)} + O((lh)^v)$$

$$L_h u_m \overset{\text{chk.}}{=} \sum_{k=0}^{v} \frac{h^k}{k!} \left(\sum_{l=p}^{q} a_l l^k \right) u_m^{(k)} + O(h^v)$$

Order v approx. of $u_m^{(n)}$ $(n \le v)$:

$$L_h u_m = u_m^{(n)} + O(h^v) \quad \text{iff} \quad \frac{h^k}{k!} \sum_{l=p}^{q} a_l l^k = \delta_{kn} = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$$

(solve for a_l, $p+q$ must be large enough)

Truncation error: $\tau_m = u_m^{(n)} - L_h u_m$

Difference operators:

Forward $\quad \Delta_h u_m = u_{m+1} - u_m$

Backward $\quad \nabla_h u_m = u_m - u_{m-1}$

Central $\quad \delta_h u_m = u_{m+\frac{1}{2}} - u_{m-\frac{1}{2}}$

$u_{m+1} = u(x_{m+1}) = u((m+1)h) = u(x_m + h)$

Rem. 1:

Shift $\quad E_h u_m = u_{m+1}, \quad \Delta_h = E_h - 1, \quad \nabla_h = 1 - E_h^{-1}, \dots$

$$E_h u_m \overset{\text{Taylor}}{=} \sum_{k=0}^{\infty} \frac{h^k}{k!} \partial^k u_m = \left(\sum \frac{h^k}{k!} \partial^k\right) u_m = e^{h\partial} u_m \quad \text{(formally)}$$

Common difference formulas:

$$u'(x_m) = \begin{cases} \dfrac{\Delta_h u_m}{h} - \frac{1}{2} h u''_m + O(h^2) & \text{F. W.} \\ \dfrac{\nabla_h u_m}{h} + \frac{1}{2} h u''_m + O(h^2) & \text{B. W.} \\ \dfrac{\delta_{2h} u_m}{2h} - \frac{1}{6} h^2 u'''_m + O(h^4) & \text{C.} \end{cases}$$

H.W. chk all formulas

$$u''(x_m) = \frac{\delta_h^2 u_m}{h^2} - \frac{1}{12} h^2 u''_m + O(h^4) \qquad \text{C}$$

Chk.: $u_{m+1} - 2u_m + u_{m-1} = \delta_h^2 u_m = \Delta_h \nabla_h u_m = (\Delta_h - \nabla_h) u_m$

Rem. 2: $\phi(h) = O(h^p),\ h \to 0 \iff \exists\, C, h_0 > 0$ s.t. $|\phi(h)| \le C h^p$ for $0 < |h| < h_0$

E. On matrices

$$A = \{a_{ij}\}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Symmetric if $A = A^T$:

(i) Real eig. val's $\{\lambda_k\}_k$ and orthonormal eig. vec's $\{\vec{x}_k\}_k$

$$A = X \Lambda X^T, \quad \Lambda = \operatorname{diag}\{\lambda_k\}, \quad X = [\vec{x}_1, \cdots, \vec{x}_n]$$

(ii) Pos. definite (semidef.) if $\vec{x}^T A \vec{x} > 0$ (≥ 0) $\forall\, \vec{x} \neq \vec{0}$

$[\iff \lambda_1, \cdots, \lambda_n > 0 \ (\ge 0)]$

(iii) Pos. def. $\Rightarrow$ invertible