Lecture 22 (Thu)

Variational problem / weak sol'n of PDE:

(V) Find $u \in V$ s.t. $a(u,v) = F(v) \quad \forall\, v \in V$

Galerkin approximation:

(V_h) Find $u_h \in V$ s.t. $a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h$

V Hilbert space, $V_h \subset V$ finitedimensional subspace

Assumptions:

(a1) $a : V \times V \to \mathbb{R}$ bilinear

(a2) a continuous (bnd.): $|a(u,v)| \le M \|u\|_V \cdot \|v\|_V \quad \forall u,v \in V \quad (M>0)$

(a3) a coersive: $a(u,u) \ge \alpha \|u\|_V^2 \quad \forall u \in V \quad (\alpha > 0)$

(F1) $F : V \to \mathbb{R}$ linear

(F2) F continuous (bnd.): $|F(v)| \le C \|v\|_V$

} $F \in V'$, $C = \|F\|_{V'} = \sup_{v \in V} \frac{|F(v)|}{\|v\|_V}$

Lax-Milgram theorem:

(assumptions) $\Rightarrow$ exists unique soln $u \in V$ of (V)

(and $u_h \in V_h$ of (V_h))

Stability:

(a3), (F2) $\Rightarrow$ $\|u\|_V \leq \frac{\|F\|_{V'}}{\alpha}$ (and $\|u_h\| \leq \frac{\|F\|_{V'}}{\alpha}$)

Galerkin orthogonality:

$$a(u - u_h, v_h) = 0 \quad \forall \; v_h \in V_h$$

Cea's lemma:

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V$$

$\longrightarrow$

Energy norm: $\|v\|_E = \langle v, v \rangle^{\frac{1}{2}}$

where i.p. $\langle u, v \rangle = \frac{1}{2}(a(u,v) + a(v,u))$

Equivalent norms:

$$\alpha^{\frac{1}{2}} \|v\|_V \underset{(a3)}{\leq} \|v\|_E \underset{(a2)}{\leq} M^{\frac{1}{2}} \|v\|_V \quad \forall \; v \in V$$

In energy norm (pf. of) Cea gives:

$$\|u - u_h\|_E \leq \inf_{v_h \in V_h} \|u - v_h\|_E$$

$\Updownarrow$

u_h is best approx. of u by $v_h \in V_h$ in $\|\cdot\|_E$

$\Updownarrow$

u_h is (orthogonal) projection of u into V_h w.r.t. $\langle \cdot, \cdot \rangle$