Lecture 3 (Tue)

A. Correction to last week

Redef. $A_h = \frac{1}{h^2}\,\text{tridiag}\{-1, 2, -1\}$

$\Rightarrow$ correct consistency and stability

See corrected note on wiki!

Rem 1: But num. we want to work w. $h^2 A_h$!

B. On matrices (cond'ed)

$$A = \{a_{ij}\}_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$, $i = 1, \ldots, n$:

(strictly DD if $\ldots >$ ...)

(i) Strictly diag. dom. $\Rightarrow$ $\lambda_i \neq 0 \ \forall i$ $\Rightarrow$ invertible

$\uparrow$ Gershgorin

(ii) not inv.: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

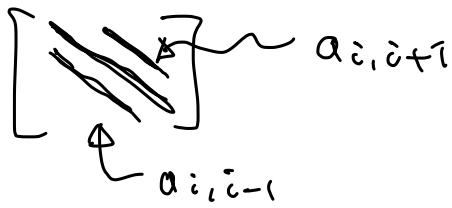
(iii) Weakly chained diag. dom. (WCDD) $\Rightarrow$ inv.

$\downarrow$

(i) diag. dom. (ii) $\exists i$ s.t. $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ (one strict row)

and (iii) "non-strict rows connected to a strict row"

Rem 2: (iii) ok if $a_{i,i+1} \neq 0$, $i = 1, \dots, n-1$ (or $a_{i,i-1} \neq 0 \dots$)



ex. $\begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ 0 & & -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & & & 0 \\ -1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & -1 & 1 \end{bmatrix}$ WCDD and inv.

Many matrices in this course are of form.

Tridiagonal if $A = \begin{bmatrix} a_1 & b_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{bmatrix}$

(i) $A = \begin{bmatrix} a & b & & 0 \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ 0 & & c & a \end{bmatrix}$ diag. w. distinct eig. val's

(I) $\lambda_k = a + 2\sqrt{bc}\cos\phi_k, \quad \phi_k = \frac{k\pi}{n+1}, \quad k = 1, \dots, n$

[HW]

(ii) If $b_i \cdot c_i > 0 \ \forall i$ then A similar to symm. matrix:

$\exists$ diag. matrix D s.t.

$$D^{-1}AD = \begin{bmatrix} a_1 & \operatorname{sgn}(b_1)\cdot\sqrt{b_1c_1} & & 0 \\ \operatorname{sgn}(b_1)\sqrt{b_1c_1} & a_2 & \operatorname{sgn}(b_2)\cdot\sqrt{b_2c_2} & \\ & \ddots & \ddots & \ddots \\ 0 & & & a_n \end{bmatrix}$$

$\operatorname{sgn} a = \begin{cases} +1, a>0 \\ -1, a<0 \end{cases}$

C. On norms

$\mathbb{R}^n$: $\vec{x} = [x_1, \dots, x_n]$

$$\|\vec{x}\|_1 = \sum_{i=1}^{n} |x_i| \,, \quad \|\vec{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \,, \quad \|x\|_\infty = \max_{i=1,\dots n} |x_i|$$

Lem. 1: $\|\vec{x}\|_2 \le \|\vec{x}\|_1 \overset{C\text{-}S}{\le} \sqrt{n}\, \|\vec{x}\|_2 \,, \quad \|\vec{x}\|_\infty \le \|\vec{x}\|_1 \le n \|\vec{x}\|_\infty$

(all norms on $\mathbb{R}^n$ are equivalent)

Subordinate matrix norms: (operator norm)

$$\|A\|_p = \sup_{\substack{\vec{x} \in \mathbb{R}^n \\ \vec{x} \neq \vec{0}}} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p} \qquad A \in \mathbb{R}^{n \times n} \,, \; p = 1, 2, \infty$$

Lem 2: $\|A\vec{x}\|_p \le \|A\|_p \cdot \|\vec{x}\|_p$; $\|A \cdot B\|_p \le \|A\|_p \cdot \|B\|_p$

$$\left[\|A \cdot B\vec{x}\|_p \overset{\text{def } \|A\|_p}{\le} \|A\|_p \cdot \|B\vec{x}\|_p \overset{\text{def } \|B\|_p}{\le} \|A\|_p \cdot \|B\|_p \|\vec{x}\|_p \overset{\text{def.}}{\Rightarrow} \|A \cdot B\|_p \le \|A\|_p \cdot \|B\|_p \right]$$

Lem 3: $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ (rows)

$\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$ (columns)

$\|A\|_2 = \sqrt{\rho(A^T A)}$, $\rho(M) = \max_{i=1,\dots,n} |\lambda_{M,i}|$ (e.v.)

spectral radius

Rem. 3:

(a) $A = A^T \Rightarrow \|A\|_2 = \rho(A)$ [diagonalise!]

(b) $\|A\| \leq \rho(A)$ for any matrix norm $\|\cdot\|$ s.t. $\|AB\| \leq \|A\| \cdot \|B\|$

[BO chp. 2]

Functions $u : \Omega \to \mathbb{R}$:

$$\|u\|_{L^1} = \int_\Omega |u(x)|\,dx, \quad \|u\|_{L^2} = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}}, \quad \|u\|_{L^\infty} = \max_{x\in\Omega} |u(x)|$$

(sup / ess sup over Ω)

Rem. 4:

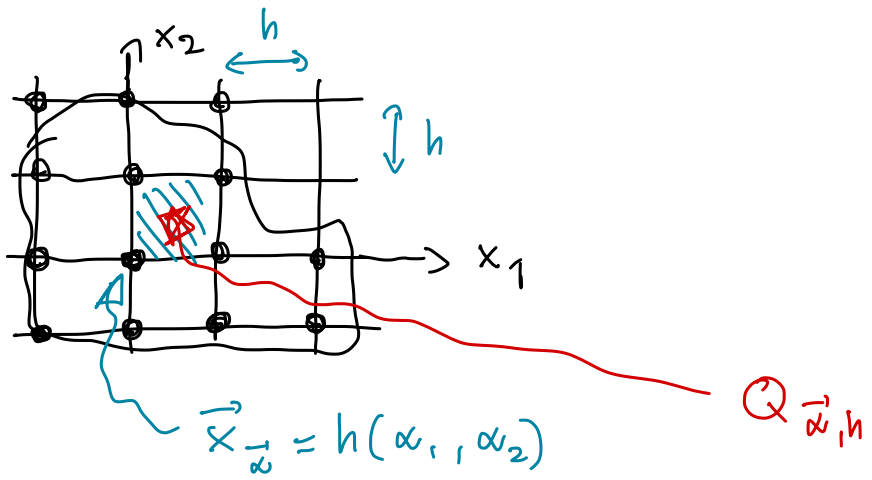
Func'n sp's: $\|\cdot\|_{L^1}, \|\cdot\|_{L^2}, \|\cdot\|_{L^\infty}$ ⇝ $L^1(\Omega), L^2(\Omega), L^\infty(\Omega)/C_b(\Omega)$

Banach sp's — bnd / cont.

D. Grid funcns $u : \Omega_h \to \mathbb{R}$

Grid for $\Omega \subset \mathbb{R}^d$: $\Omega_h \subset h \cdot \mathbb{Z}^d = \{ h \cdot \vec{\alpha} : \vec{\alpha} \in \mathbb{Z}^d \}$

$= \vec{x}_{\vec{\alpha}}$



Divides Ω into cubes

$$Q_{\vec{\alpha},h} = h\vec{\alpha} + h^d [0,1)^d \quad ; \quad \text{Vol}(Q_h) = h^d$$

$= [0,h)^d$

Indexing: Multi-index $\vec{\alpha} \in \frac{1}{h}\Omega_h \subset \mathbb{Z}^d$ or index $i = e(\vec{\alpha})$

where $e : \frac{1}{h}\Omega_h \to \mathbb{N}$ is a (1-1) enumeration of Ω_h

Def.:

$$u_{\vec{\alpha}} = u(h\vec{\alpha}) = u(\vec{x}_{\vec{\alpha}}) \quad \text{and} \quad u_i = u_{\vec{\alpha}} \quad \text{when} \quad i = e(\vec{\alpha})$$

Obs 1: $\vec{U} = \{U_i\}_{i=1}^{M_h} \in \mathbb{R}^{M_h}$, where

M_h = no. of grid pt's in Ω_h $\quad (< \infty, \xrightarrow[h \to 0]{} \infty)$

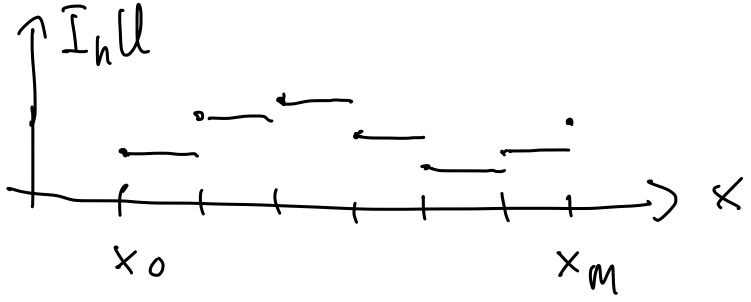
Interpolant $I_h U$:

$$I_h U : \Omega \to \mathbb{R} \quad \text{s.t.} \quad I_h U_{\vec{\alpha}} = U_{\vec{\alpha}} \quad \forall\ \vec{\alpha} \in \tfrac{1}{h}\Omega_h$$

P.w. const. (PO) interp.:

$$I_h U(x) = U_{\vec{\alpha}} \quad \text{for} \quad x \in Q_{\vec{\alpha},h} \quad \forall\ \vec{\alpha}$$

1d:



Rem. 5:

(a) Interpolation of U gives approx. on all of Ω! (not only on Ω_h)

(b) Conv. of $U_h \to u$ can (should!) be interpret as

$$I_h U_h \longrightarrow u \quad \text{in} \quad \Omega \quad (\text{or } L^p(\Omega) \text{ or } C_b(\Omega))$$

and then

$$\|u - I_h U_h\|_{L^p} \le \underbrace{\|u - I_h u\|_{L^p}}_{\text{interp. error}} + \underbrace{\|I_h(u - U_h)\|_{L^p}}_{\text{interp. discr. error (a num. quadrature)}}$$

(c) $I_h = P0$ interp.:

(i) $\|u - I_h u\|_{L^p} = O(h)$; or $O(h^2)$ if $\vec{x}_{\vec{\alpha}}$ midpt. in $Q_{\vec{\alpha},h}$ (!)

(ii) $$\|I_h(u - U_h)\|_{L^p(\Omega)} = \left(\sum_{\vec{\alpha}} \int_{Q_{\vec{\alpha},h}} |I_h(u - U_h)|^p \right)^{\frac{1}{p}}$$

$$\Omega = \bigcup_{\vec{\alpha}} Q_{\vec{\alpha},h}$$

$$= \left(\sum_{\vec{\alpha}} |u_{\vec{\alpha}} - U_{h,\vec{\alpha}}|^p h^d \right)^{\frac{1}{p}}$$

$$= \left(\sum_{i} |u_i - U_{h,i}|^p \right)^{\frac{1}{p}} h^{\frac{d}{p}}$$

$$= \|\vec{u} - \vec{U}_h\|_{p, \mathbb{R}^{M_h}} \cdot h^{\frac{d}{p}}$$

$$\| I_h(u-U_h) \|_{L^\infty(\Omega)} = \| \vec{u} - \vec{U}_h \|_{\infty, \mathbb{R}^{M_h}}$$

Obs. 2: $\Omega = \bigcup_{\vec{\alpha}_i} Q_{\vec{\alpha}_i, h} \subset \mathbb{R}^d$, $p \in [1, \infty]$, I_h PO interp

$$\Rightarrow \| \vec{e}_h \|_p \cdot h^{\frac{d}{p}} = \| I_h e_h \|_{L^p} \qquad \left(\frac{d}{\infty} = 0 \right)$$

E. Stability and conv. in 2-norm

(P) $-u_{xx} = f$ in $(0,1)$; $u(0) = g_0$, $u(1) = g_1$

(P_h) $-\frac{1}{h^2} \delta_h^2 U_m = f_m$, $m = 1, \dots, M-1$; $U_0 = g_0$, $U_M = g_1$; $h = \frac{1}{M}$

($\tilde{P}_h$) $A_h \vec{U} = \vec{F}$; $A_h = \frac{1}{h^2} \operatorname{tridiag}\{-1, 2, -1\}$; $\vec{F} = [f_1 + \frac{g_0}{h^2}, f_2, \dots, f_{M-1} + \frac{g_1}{h^2}]$

A_h: $(M-1) \times (M-1)$; $\vec{F}$: $(M-1)$

Rem. 6:

a) $h^2 A_h$ has eig.-val's ($a = 2$, $b = c = -1$)

$$\lambda_k \overset{(1)}{=} a + 2\sqrt{bc} \cdot \cos \frac{k\pi}{M+1} = 2\left(1 + \cos \frac{k\pi}{M+1}\right), \quad k = 1, \dots, M$$

$$\lambda_{\min} \overset{\text{chk}}{=} 2\left(1 + \cos \frac{M\pi}{M+1}\right) \overset{\text{chk.}}{=} 2\left(1 - \cos \frac{\pi}{M+1}\right) \overset{\substack{\text{Taylor} \\ \text{chk}}}{=} 2 \cdot \frac{1}{2} \left(\frac{\pi}{M+1}\right)^2 + O\left(\frac{1}{M^4}\right)$$

$$\Rightarrow \lambda_{A_h,\min} = \frac{1}{h^2}\lambda_{\min} \overset{\text{chk.}}{=} \pi^2 + O(h^2) \qquad [\, h = \frac{1}{M} \,]$$

b) $\lambda_{A_h,k} > \lambda_{A_h,\min} > 0$, h small enough $\Rightarrow A_h$ inv.

c) $\|A_h^{-1}\|_2 \overset{A_h^{-1}\text{ symm.}}{=} \rho(A_h^{-1}) = \max\left|\frac{1}{\lambda_{A_h}}\right| = \frac{1}{|\lambda_{A_h,\min}|} = \frac{1}{\pi^2} + O(h^2)$

Lecture 4 (Thu)

A. Stability and conv. in 2-norm

(P) $-u_{xx} = f$ in $(0,1)$; $u(0) = g_0$, $u(1) = g_1$

(P_h) $-\frac{1}{h^2}\delta_h^2 U_m = f_m$, $m = 1, \dots, M-1$; $U_0 = g_0$, $U_M = g_1$; $h = \frac{1}{M}$

($\tilde{P}_h$) $A_h \vec{U} = \vec{F}$; $A_h = \frac{1}{h^2}\,\text{tridiag}\{-1, 2, -1\}$; $\vec{F} = [f_1 + \frac{g_0}{h^2}, f_2, \dots, f_{M-1} + \frac{g_1}{h^2}]$

A_h: $(M-1)\times(M-1)$; $\vec{F}$: $(M-1)$

Last time: $\|A_h^{-1}\|_2 = \frac{1}{|\lambda_{A_h,\min}|} = \frac{1}{\pi^2} + O(h^2)$

2-norm stability:

$A_h \vec{V} = \vec{g}$, $V_0 = 0 = V_M \Rightarrow \|\vec{V}\|_2 \le \|\vec{g}\|_2$ for h small enough

$[\|\vec{V}\|_2 = \|A_h^{-1}\vec{g}\|_2 \le \|A_h^{-1}\|_2 \|\vec{g}\|_2 \le 1 \cdot \|g\|_2$, h small enough$]$

Consistency:

$$\tau_{h,m} = (u_{xx})_m - \frac{1}{h^2}\delta_h^2 u_m = \frac{1}{12}h^2 u^{(4)}(\xi_m)$$

Obs: $\vec{x} = (x_1, \dots, x_n)$, $\|\vec{x}\|_2 \le \sqrt{n}\,\|\vec{x}\|_\infty$

$$\|\vec{\tau}_h\|_2 \le \frac{1}{12}h^2 \|\vec{u}^{(4)}(\xi_\cdot)\|_2 \le \frac{1}{12}h^2\sqrt{M}\cdot\|\vec{u}^{(4)}(\xi_\cdot)\|_\infty \overset{h=\frac{1}{M}}{\le} \frac{1}{12}h^{\frac{3}{2}}\|u^{(4)}\|_{L^\infty} \quad (\square)$$

↳ can not be improved using $\|u^{(4)}\|_{L^2}$ argument (error on p 18 in BO)

Error eq'n: $e_{h,m} = u_m - U_m$

$$(2)\quad A_h\vec{e} = A_h\vec{u} - A_h\vec{U} \overset{(P_h)}{=} A_h\vec{u} - \frac{1}{h^2}\begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix} - \vec{f} \overset{(P)}{=} -\frac{1}{h^2}\delta_h^2\vec{u} + \vec{u}_{xx} = \vec{\tau}_h$$

$$\vec{F} = \vec{f} + \frac{1}{h^2}\begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix} \quad \text{clik.} = -\frac{1}{h}\delta_h^2\vec{u} \qquad -\vec{u}_{xx}$$

Convergence: h small enough

$$\|\vec{e}_h\|_2 \overset{(2)}{\underset{\text{stab}}{\le}} \|\vec{\tau}_h\|_2 \le \frac{1}{12}\|u^{(4)}\|_{L^\infty}h^{\frac{3}{2}}$$

↖ this is correct

Rem 7:

a) $\|u^{(4)}\|_{L^\infty} \le \|f''\|_{L^\infty}$

b) $\|I_h e_h\|_{L^2} \le h^{\frac{1}{2}}\|\vec{e}_h\|_2 \le \frac{1}{12}\|u^{(4)}\|_{L^\infty}h^2$ [2^{nd} order meth. in L^2!] This is optimal!

c) $\|\vec{e}_h\|_2 \le \sqrt{M}\cdot\|\vec{e}_h\|_\infty \overset{\text{last week}}{\le} \frac{1}{96}\|f''\|_\infty h^2\cdot\sqrt{M} = O(h^{\frac{3}{2}})$ (similar to $(\square)$)

B. Neumann and mixed BVP

Dirichlet B.C. : u given at $\partial\Omega$

Neumann B.C. : $\frac{\partial u}{\partial n}$ — " —

Mixed B.C. : combination

1d mixed BVP:

$$(M) \quad -u_{xx} = f(x) \text{ in } (0,1) \; ; \quad u_x(0) = \sigma \; , \; u(1) = \beta$$

Discretise PDE and Dir. B.C. :

$$(M_h) \quad -\frac{1}{h^2}\delta_h^2 U_m = f_m \; , \; m = 1, \dots, M-1 \; ; \quad U_M = \beta \; , \; h = \frac{1}{M}$$

Alternative discr. of Neumann B.C:

1) $u_x(0) = \frac{u_1 - u_0}{h} + O(h) \quad \Rightarrow \quad \frac{U_1 - U_0}{h} = \sigma$

2) $u_x(0) = \frac{-3u_0 + 4u_1 - u_2}{2h} + O(h^2) \quad \Rightarrow \quad \frac{-3U_0 + 4U_1 - U_2}{2h} = \sigma$

3) Fictitious node: x_{-1}

$$u_x(0) = \frac{u_1 - u_{-1}}{2h} + O(h^2) \quad \Rightarrow \quad \frac{U_1 - U_{-1}}{2h} = \sigma \qquad (*)$$

Extra unknown u_{-1}, need extra eq'n (take interior eq'n at $m=0$):

$$(**) \quad f_0 \overset{(M_h')}{=} -\frac{1}{h^2}\delta_h^2 u_0 = \frac{-u_{-1}+2u_0-u_1}{h^2}$$

Eliminate u_{-1}:

$$(*) \Rightarrow u_{-1} = u_1 - \sigma 2h$$

$$(**) \Rightarrow \text{chk!} \quad \boxed{\frac{u_1-u_0}{h} = -\frac{f_0 h}{2} + \sigma}$$

Case 3):

$$(M_h) \quad (M_h') + \frac{-2u_1+2u_0}{h^2} = f_0 - \frac{2\sigma}{h} \quad (\Leftrightarrow 3)\cdot\left(-\tfrac{2}{h}\right))$$

$$\Updownarrow$$

$$(\tilde{M}_h) \quad \underbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -2 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}}_{A_h\ (M\times M)} \cdot \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{M-1} \end{bmatrix}}_{\vec{u}\ (M)} = \underbrace{\begin{bmatrix} f_0 - \frac{2\sigma}{h} \\ f_1 \\ \vdots \\ f_{M-1} - \frac{\beta}{h^2} \end{bmatrix}}_{\vec{F}\ (M)}$$

Rem 2:

1) 1st order

2) 2nd order, A_h no longer tridiag. or symm. !

3) 2nd order, A_h tridiag., but not symm.

Rem 3: a) A_h in case 3) is WCDD and inv. !

b) (M_h) mon. meth. $\left[(a_h)_{ii} > 0 \,,\ (a_h)_{ij} \underset{j \neq i}{\leq} i \,,\ \text{diag-dom.} \right] \Rightarrow$ discr. max. pr. holds ($\Leftrightarrow$ L^∞ stability)

C. Pure Neumann BVP

$$(N) \qquad -u_{xx} = f(x) \ \text{ in } (0,1) \,; \quad u_x(0) = \sigma_0 \,, \quad u_x(1) = \sigma_1$$

No uniqueness: u sol'n $\Rightarrow$ $u + C$ sol'n $\forall\, C \in \mathbb{R}$

Not always ex.:

$$u \text{ solves } (N) \Rightarrow \boxed{\int_0^1 f(x)\,dx} = -\int_0^1 u_{xx}\,dx = -\left(u_x(1) - u_x(0)\right) = \boxed{\sigma_0 - \sigma_1} \qquad (***)$$

($\Leftrightarrow$)

$(***)$ not holds $\Rightarrow$ no u solves (N) !

Type 3) discr.:

$$(N_h)\quad A_h\vec{u} = \vec{F}, \qquad A_h = \begin{bmatrix} 2 & -2 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -2 & 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} u_0 \\ \vdots \\ u_M \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} f_0 - \frac{2\sigma_0}{h} \\ f_1 \\ \vdots \\ f_M - \frac{2\sigma_1}{h} \end{bmatrix}$$

$(M+1)\times(M+1)$ $\qquad$ $(M+1)$ $\qquad$ $(M+1)$

Not uniq.: $\vec{u}$ soln $\Rightarrow \vec{u} + c\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ soln $\forall\, c \in \mathbb{R}$

Not always ex.: $A_h \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \Rightarrow \lambda = 0$ eig. val. $\Rightarrow A_h$ not inv.

Ex. under discrete verson of (***) cond'n, see BO sec. 3.1

Rem 4:

a) Add cond'n to get uniq. soln: $\int_0^1 u\,dx = 0$ or $u(\frac{1}{2}) = 0$ or ...

b) $-u_{xx} + bu = f$; $u_x(0) = \sigma_0$, $u_x(1) = \sigma_1$ $\Longrightarrow$ ex. uniq. soln when $b > 0$!

D. BVPs for gen. lin. ODEs

$$(GL)\quad -u_{xx} + p(x)u_x + q(x)u = f(x) \quad \text{in } (a,b)\,; \quad u(a) = g_0\,, \quad u(b) = g_1$$

Rem 5: a) Ex. unq. sol'n of (GL) if e.g.

$$p, q, f \in C[a,b] \text{ and } q > 0 \text{ in } [a,b]$$

b) $-a(x)u_{xx} + \dots = f(x)$; $a(x) \geq a_0 > 0$ $\iff$ $\cdot\frac{1}{a}$ $-u_{xx} + \dots = \frac{f(x)}{a(x)}$

Discretisation:

$$-\frac{u_{m+1} - 2u_m + u_{m-1}}{h^2} + p(x_m)\frac{u_{m+1} - u_{m-1}}{2h} + q(x_m)u_m = f_m$$

$\Updownarrow \cdot h^2$

$$\left(-1 + \frac{h}{2}p_m\right)u_{m+1} + \left(2 + h^2 q_m\right)u_m + \left(-1 - \frac{h}{2}p_m\right)u_{m-1} = h^2 f_m$$

$\Updownarrow \cdot \frac{1}{h^2}$

$$(GL_h) \quad A_h \vec{u} = \vec{F}, \quad A_h = \frac{1}{h^2}\begin{bmatrix} 2+h^2q_1 & -1+\frac{h}{2}p_1 & & 0 \\ -1-\frac{h}{2}p_2 & 2+h^2q_2 & -1+\frac{h}{2}p_2 & \\ & \ddots & \ddots & \ddots \\ 0 & & & 2+h^2q_{M-1} \end{bmatrix}$$

Rem. 6:

(a) A_h not symm. if $p \neq 0$

(b) A_h str. diag. dom. and inv. if

$$(\Delta) \quad |2 + h^2 q_m| > |-1 - \tfrac{h}{2}p_m| + |-1 + \tfrac{h}{2}p_m|, \quad m = 1, \dots, M-1$$

(Δ) ok if e.g.:

(ΔΔ) $q_m \geq 0$ and $\frac{1}{2} h |p_m| < 1$, $m = \dots$ (chk!)

$\Uparrow$

$q \geq 0$ and $h < \frac{2}{\|p\|_{L^\infty}}$

(c) (ΔΔ) $\Rightarrow (a_h)_{ii} > 0$, $(a_h)_{ij} \underset{j \neq i}{\leq} 0$, $(a_h)_{ii} \geq \sum_{j \neq i} |(a_h)_{ij}|$

$\Rightarrow$ (GLh) pos. coeff. / mon. $\Rightarrow$ discr. max. pr. holds ($\Rightarrow L^\infty$ stab.)

E. Self-adjoint problems

(SA) $-(K(x) u_x)_x = f(x)$ in (a,b); $u(a) = g_0$, $u(b) = g_1$

where $f \in L^2(a,b)$, $K \in C[a,b]$, $K(x) \geq K_0 > 0$ for $x \in [a,b]$

Rem. 7:

(a) $\mathcal{L} \varphi(x) = -(K(x)\varphi_x)_x$, $\mathcal{L}$ self-adjoint in L^2:

$\langle \mathcal{L}\varphi, \psi \rangle = \langle \varphi, \mathcal{L}\psi \rangle \quad \forall\, \varphi, \psi \in C_0^2[a,b]$

$\Rightarrow (\varphi(a) = 0 = \varphi(b))$

where L^2-i.p. $\langle \varphi, \psi \rangle = \int_a^b \varphi \cdot \psi \, dx$.

(b) $(SA)\cdot v + \int_a^b dx \xrightarrow[v(a)=0=v(b)]{\text{1 x i.b.p}} \underbrace{\int_a^b u_x v_x K\,dx}_{\text{an i.p. in } H_0^1(a,b)} = \underbrace{\int_a^b f v\,dx}_{\text{lin. func'al on } H_0^1 \text{ (and } L^2)}$

Riesz repr. thm. $\Rightarrow$ ex. uniq. sol'n $u \in H_0^1(a,b)$ $\quad [\langle u,v\rangle_{H_0^1} = B(v) \;\forall\, v \in H_0^1]$

Discretisation:

1) $(SA) \Leftrightarrow -Ku_{xx} - K_x u_x = f \rightsquigarrow$ discretise

Bad: Result not symm./self-adj.: $\langle A_h \vec{u}, \vec{V}\rangle_{\mathbb{R}^{M-1}} \neq \langle \vec{u}, A_h \vec{V}\rangle_{\mathbb{R}^{M-1}}$

(also: need K_x – a problem if K not diff.)

2) Direct discr.:

$$(Ku_x)_x = \frac{1}{h}\delta_h\left(K\frac{1}{h}\delta_h u\right) + O(h^2) \qquad (\text{if } K \in C^2[a,b])$$

$$\delta_h(K\delta_h u)_m = (K\delta_h u)_{m+\frac12} - (K\delta_h u)_{m-\frac12} = K_{m+\frac12}(u_{m+1}-u_m) - K_{m-\frac12}(u_m - u_{m-1})$$

Scheme:

$$(SA_h) \quad -\frac{1}{h^2}\delta_h(K\delta_h u)_m = f_m,\ m = 1,\dots,M-1;\ u_0 = g_0,\ u_M = g_1$$

$$\Updownarrow$$

$$A_h\vec{u} = \vec{F}, \quad A_h = \frac{1}{h^2}\begin{bmatrix} K_{\frac12}+K_{\frac32} & -K_{\frac32} & 0 \\ -K_{\frac32} & K_{\frac32}+K_{\frac52} & -K_{\frac52} \\ & \ddots & \ddots \end{bmatrix}, \dots$$

Consistency:

$$\tau_{h,m} = (u_{xx})_m - \frac{1}{h^2}\delta_h^2 u_m = \frac{1}{12}h^2 u^{(4)}(\xi_m)$$

$$\|\vec{\tau}_h\|_2 \le \frac{1}{12}h^2 \|u^{(4)}\|_{L^\infty}\left(\sum_{i=1}^{m-1} 1^2\right)^{\frac{1}{2}} \le \frac{1}{12}h^2\|u^{(4)}\|_{L^\infty}\cdot \underbrace{\sqrt{M}}_{\| \frac{1}{\sqrt{h}}} = O(h^{\frac{3}{2}})$$

← come back to it

Error eqn: $e_{h,m} = u_m - U_m$

$$(2)\quad A_h\vec{e} = A_h\vec{u} - A_h\vec{U} \overset{(P_h)}{=} A_h\vec{u} - \frac{1}{h^2}\begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix} - \vec{f} \overset{(P)}{=} -\frac{1}{h^2}\delta_h^2\vec{u} + \vec{u}_{xx} = \vec{\tau}_h$$

$$\vec{F} = \vec{f} + \frac{1}{h^2}\begin{bmatrix} g_0 \\ 0 \\ \vdots \\ g_1 \end{bmatrix} \quad \text{clik.} = -\frac{1}{h}\delta_h^2\vec{u} \qquad -\vec{u}_{xx}$$

Convergence: h small enough

$$\|\vec{e}_h\|_2 \overset{(2)}{\underset{\text{Stab}}{\le}} \|\vec{\tau}_h\|_2 \le \frac{1}{12}\|u^{(4)}\|_{L^\infty} h^{\frac{3}{2}}$$

Rem 7: a) $\|u^{(4)}\|_{L^\infty} \le \|f''\|_{L^\infty}$

b) $\|I_h e_h\|_{L^2} \le h^{\frac{1}{2}}\|\vec{e}_h\|_2 \le \frac{1}{12}\|u^{(4)}\|_{L^\infty} h^2$

c) $\|\vec{e}_h\|_2 \le \sqrt{M}\cdot\|\vec{e}_h\|_\infty \overset{\text{last week}}{\le} \frac{1}{96}\|f''\|_\infty h^2\cdot\sqrt{M} = O(h^{\frac{3}{2}})$