# Lecture 5 (Tue)

## A. Self-adjoint problems

$L^2$ - self adjoint problem

(SA) $\quad -(K(x)u_x)_x = f(x) \text{ in } (a,b); \quad u(a)=g_0,\ u(b)=g_1$

where $\quad f \in L^2(a,b),\ K \in C[a,b],\ K(x) \geq K_0 > 0 \text{ for } x \in [a,b]$

Discretization:

$$(SA_h) \quad -\frac{1}{h^2}\delta_h(K\delta_h U)_m = f_m,\ m=1,\dots,M-1;\ U_0=g_0,\ U_M=g_1$$

$$\Updownarrow$$

$$A_h\vec{U} = \vec{F}, \quad A_h = \frac{1}{h^2}\begin{bmatrix} K_{\frac12}+K_{\frac32} & -K_{\frac32} & & 0 \\ -K_{\frac32} & K_{\frac32}+K_{\frac52} & -K_{\frac52} & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \end{bmatrix}$$

Good: (i) $A_h = A_h^T$, $A_h$ symm. and $(SA_h)$ self adj.

(ii) $A_h$ is WCOD and inv. (chk, $K>0$)

(iii) $(SA_h)$ monotone/pos. coeff.: $(a_h)_{ii} > 0,\ (a_h)_{ij} \leq 0,\ j \neq i;\ (a_h)_{ii} \geq \sum_{j\neq i}(a_h)_{ij}$

(iv) Only need $K$, never $K_x$ !

# Lecture 6 (Thu)

## A. Num. sol'n of heat eq'n

(H) $u_t = u_{xx}$ in $(0,1)\times(0,T)$ ; $u(0,t) = g_0(t)$ , $u(1,t) = g_1(t)$ ; $u(x,0) = u_0(x)$

Grid: $(x_m, t_n) = (m\cdot h, n\cdot k)$ ; $h = \frac{1}{M}$ , $k = \frac{T}{N}$ ; $U_m^n \simeq u_m^n := u(x_m, t_n)$

"Forward Euler":

(FE) $\frac{1}{k}\Delta_t U_m^n = \frac{1}{h^2}\delta_x^2 U_m^n$ , $m = 1, \dots, M-1$ ; $U_0^n = g_0(t_n)$, $U_M^n = g_1(t_n)$ ; $U_m^0 = u_0(x_m)$, $m = 0, \dots, M$

$\Updownarrow \quad r = \frac{k}{h^2}$

$U_m^{n+1} = r U_{m-1}^n + (1-2r) U_m^n + r U_{m+1}^n$ + B.C. $\iff \vec{U}^{n+1} = B\vec{U}^n + \vec{b}^n$ (B.C.), $B = \text{tridiag}\{r, 1-2r, r\}$, $\vec{b}^n = [r g_0(t_n), 0, \dots, 0, r g_1(t_n)]$

"Backward Euler":

(BE) $\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{h^2}\delta_x^2 U_m^{n+1}$, $m = 1, \dots, M-1$ ; ...

$\Updownarrow \quad r = \frac{k}{h^2}$

$-r U_{m-1}^{n+1} + (1+2r) U_m^{n+1} - r U_{m+1}^{n+1} = U_m^n$ + B.C. $\iff A\vec{U}^{n+1} = \vec{U}^n + \vec{b}^{n+1}$ , $A = \text{tridiag}\{-r, 1+2r, -r\}$

Rem. 1:

(a) Consistency of (FE) and (BE): $\tau_h = O(k + h^2)$

(b) Stability:

(FE) stable when $\max |\lambda_B| < 1 \iff r = \frac{k}{h^2} \leq \frac{1}{2}$ CFL-cond'n

Conditional stable: small time steps $k \leq \frac{1}{2} h^2$

(BE) stable when $\max |\lambda_{A^{-1}}| = \frac{1}{\min |\lambda_A|} < 1$, always true!

Unconditionally stable

(c) (BE) implicit: At each time step we must solve a (lin) syst. to find $\vec{u}^{n+1}$

(FE) explicit: not implicit

(d) (FE) monotone when $k < \frac{1}{2} h^2$, (BE) always mon.