Lecture 7 (Tue)

A. Forward and backward Euler scheme for the heat eq'n

(H) $u_t = u_{xx}$ in $(0,1)\times(0,T)$; $u(0,t) = g_0(t)$, $u(1,t) = g_1(t)$; $u(x,0) = u_0(x)$

(FE) $\frac{1}{k}\Delta_t U_m^n = \frac{1}{h^2}\delta_x^2 U_m^n$ + B.C./I.C. $\overset{r=\frac{k}{h^2}}{\Longleftrightarrow}$ $\vec{U}^{n+1} = B\vec{U}^n + \vec{b}^n$, $B = \text{tridiag}\{r, 1-2r, r\}$, $\vec{b}^n$ B.C. — explicit

(BE) $\frac{1}{k}\nabla_t U_m^{n+1} = \frac{1}{h^2}\delta_x^2 U_m^{n+1}$ + B.C./I.C. $\Longleftrightarrow$ $A\vec{U}^{n+1} = \vec{U}^n + \vec{b}^{n+1}$, $A = \text{tridiag}\{-r, 1+2r, -r\}$ — implicit

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Stability: (FE) stable if $\|\vec{V}^n\| \le C \max_{n=0,\dots,N} \|\vec{f}^n\|$, C indep. of h, k and $\vec{V}$ solves

(FE_f) $\vec{V}^{n+1} = B\vec{V}^n + k\vec{f}^n$, $V_0^n = V_M^n = 0$, $\vec{V}^0 = \vec{0}$.

| similar defi. for (BE) with (BE_f)...

Thm 1: $r = \frac{k}{h^2} \le \frac{1}{2}$ (CFL-cond'n) $\Longleftrightarrow$ (FE) stable for $\|\cdot\| = \|\cdot\|_p$, $p = 1, 2, \infty$. ← conditionally stable

Thm 2: (BE) always stable for $\|\cdot\| = \|\cdot\|_p$, $p = 2, \infty$. ← uncond. stable

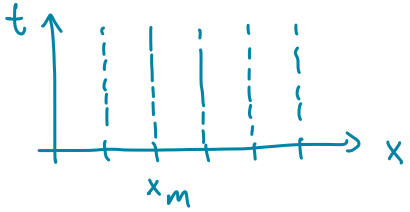
Rem.: In all cases $\|\vec{V}^n\|_p \le T \max_{n=0,\dots,N} \|\vec{f}^n\|_p$, where $\vec{V}^n$ solves (FE_f) or (BE_f)

Lecture 8 (Thu)

General Parabolic PDEs, the method of lines (MOL)

$$(P)\quad u_t = F(t, x, u, u_x, u_{xx}) \text{ in } (0,1)\times(0,T) \quad + \text{B.C., I.C. } u(x,0) = u_0(x)$$

(A) Spatial grid: $x_m = m\cdot h$, $h = \frac{1}{M}$



(B) Semi-discretise (only space)

$$u_x(x_m,t) = \frac{1}{h}\begin{Bmatrix}\Delta_x\\ \nabla_x\\ \frac{1}{2}\delta_{2h,x}\end{Bmatrix} u_m(t), \qquad u_{xx}(x_m,t) = \frac{1}{h^2}\delta_x^2 u_m(t), \qquad v_m(t) \simeq u(x_m,t) =: u_m(t)$$

$$\Downarrow (P)$$

$$(P_h)\quad \dot v_m(t) = F\left(t, x_m, v_m, \frac{1}{h}\begin{Bmatrix}\Delta_x\\ \vdots\end{Bmatrix} v_m, \frac{1}{h^2}\delta_x^2 v_m\right), \quad m = 1,\dots,M-1 \quad + \text{B.C.}, \qquad v_m(0) = u_0(x_m)$$

(C) Discretise in time: Use favourite ODE solver

Example: Trapezoidal rule → Crank-Nicholson

$$u_t = u_{xx} \longrightarrow U_m^{n+1} = U_m^n + \frac{1}{2}\frac{k}{h^2}\left[\delta_x^2 U_m^n + \delta_x^2 U_m^{n+1}\right]$$

$O(k^2+h^2)$, implicit, mon. if $r = \frac{k}{h^2} \le 1$