Lecture 9 (Tue)

Motivation for von Neumann stability analysis

$$u_t = u_{xx} \quad \text{in } (0,1)\times(0,T), \quad u(x,0) = u_0(x), \quad \text{periodic B.C. } u(0,t) = u(1,t)$$

Fourier-anal. (Mat 4):

$$u_0(x) = \sum_{l=-\infty}^{\infty} c_l e^{i\beta_l x}, \quad \beta_l = 2\pi l, \quad c_l = \int_0^1 u_0(x) e^{-i\beta_l x}\, dx$$

Separation of variables (Mat 4):

$$u_l(x,t) = e^{-\beta_l^2 t} \cdot e^{i\beta_l x} \quad \text{solves} \quad \text{PDE} + \text{per. B.C.}$$

Superposition (Mat 4):

$$u(x,t) = \sum u_l = \sum_{l=-\infty}^{\infty} c_l e^{-\beta_l^2 t} \cdot e^{i\beta_l x} \quad \text{solves} \quad (P)$$

Parseval:

$$(*) \quad \sum_{l=-\infty}^{\infty} |c_l|^2 = \int_0^1 u_0(x)^2\, dx$$

$$\Rightarrow \int |u(x,t)|^2\, dx \overset{(*)}{=} \sum |e^{-\beta_l^2 t} c_l|^2 < \sum |c_l|^2 \overset{(*)}{=} \int_0^1 u_0(x)^2\, dx$$

Lecture 10 (Thu)

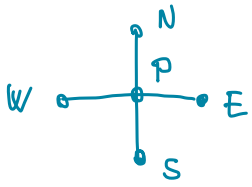
Model problem: Poisson on $[0,1]^2$

$$(P) \quad -\Delta u = -u_{xx} - u_{yy} = f(x,y) \quad \text{in} \quad \Omega = [0,1]^2, \qquad u = g \quad \text{in} \quad \partial\Omega$$

Grid: $\overline{\mathbb{G}} = \{P_{ij}\}_{ij} = \mathbb{G} \cup \partial\mathbb{G}$, interior ∪ boundary grid pt's

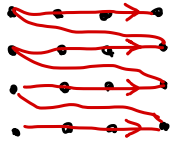
$$P = P_{ij} = (x_i, y_j) = (ih, jh), \quad h = \frac{1}{M}, \quad u_p = u_{ij} \simeq u_{ij} := u(x_i, y_j) = u(P) =: u_p$$

FDM:



$$(P_h) \quad -\mathcal{L}_h u_p = -\frac{1}{h^2}(\delta_x^2 + \delta_y^2)u_p = \frac{1}{h^2}(-u_E - u_W - u_N - u_S + 4u_p) = f(P) \quad \forall\, P \in \mathbb{G}$$

Natural/row-wise ordering:



$$u_{i,j} = u_{i+(M-1)j}, \qquad \vec{u} = [u_1, u_2, \ldots, u_{(M-1)^2}]^T$$

$$(P_h) \iff \underset{(M-1)^2 \times (M-1)^2}{A_h}\,\vec{u} = \vec{b}, \qquad A_h = \text{tridiag}\{-I, B, -I\}, \quad B = \text{tridiag}\{-1, 4, -1\}, \quad I = \text{diag}\{1\}$$

A_h: block-tridiag., sparse, inv. (WCOD), mon. $[a_{ii} > 0, \; a_{ij} \overset{i\neq j}{\leq} 0, \; a_{ii} \geq \sum_{j\neq i} |a_{ij}|]$