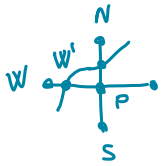
Lecture 11 (Tue)

General domains

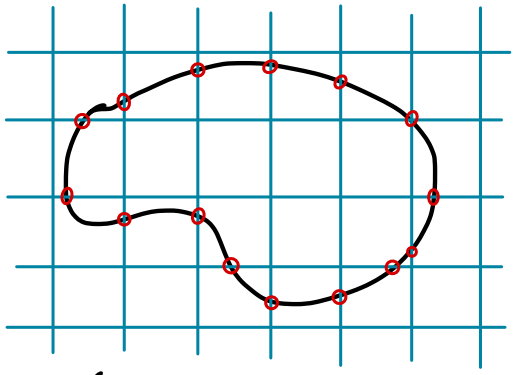
I Modify FDMs near bnd'ry

Add bnd'ry nodes to grid (irregular grid)

Modify FDMs near bnd'ry:



$$u_{xx} = a_{W'} u_{W'} + a_P u_P + a_E u_E + \tau_P \xrightarrow{\text{chk}} a_{W'} = \frac{2}{h^2}\cdot\frac{1}{\eta(1+\eta)},\ a_E = \frac{2}{h^2}\cdot\frac{1}{1+\eta},\ a_P = \frac{2}{h^2}\cdot\frac{1}{\eta},\ \tau = O(h)$$

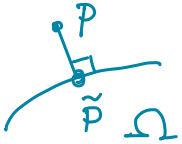


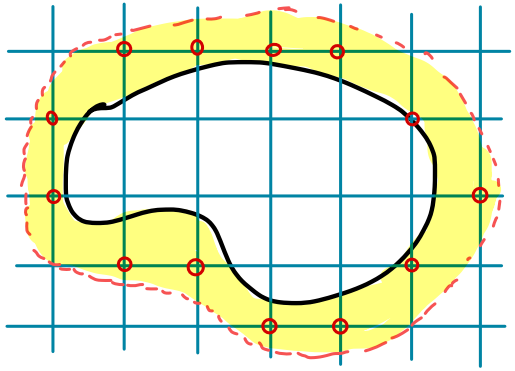
II Fatten the boundary, extend B.C.

Add nodes in fattened bnd'ry to grid (regular grid)

Extend B.C.'s:

$$u_P = g(\tilde{P})$$





where $\tilde{P}$ closest bnd'ry pt. to P : $|P-\tilde{P}| = \min\{|Q-P| : Q \in \partial G\}$

Error: Ext. u smoothly to P, then

$$e_P = u_P - U_P = \underbrace{u_{\tilde{P}}}_{g(\tilde{P})} + \underbrace{(P-\tilde{P})}_{<\sqrt{2}\cdot h} u_x(\tilde{P}) - g(\tilde{P}) = O(h)$$

Case I and II:
ONLY 1st ORDER NEAR $\partial\Omega$

Lecture 12 (Thu)

Monotone discretisations

$$-\mathcal{L}_h V_P = \alpha_{PP} V_P - \sum_{Q \neq P} \alpha_{PQ} V_Q, \qquad P \in \mathbb{G}, \quad Q \in \overline{\mathbb{G}}, \qquad \overline{\mathbb{G}} = \mathbb{G} \cup \partial\mathbb{G} \text{ (possibly irregular) grid}$$

Monotone / pos. coeff. if:

(A1) $\alpha_{PP} > 0, \quad \alpha_{PQ} \geq 0, \quad \alpha_{PP} \geq \sum_{Q \neq P} \alpha_{PQ}, \qquad P \in \mathbb{G}$

diag. dom.

Discrete max. principle

$\mathbb{G}$ boundary connected if:

(A2) For each $P \in \mathbb{G}$, there is $Q \in \partial\mathbb{G}$ and $P_1, \dots, P_k \in \mathbb{G}$ s.t. $\alpha_{PP_1}, \alpha_{P_1P_2}, \dots, \alpha_{P_kQ} \neq 0$

Theorem: Assume (A1) and (A2), then

(DMP) $-\mathcal{L}_h V_P \leq 0$ in $\mathbb{G} \implies \max_{P \in \mathbb{G}} V_P \leq \max_{P \in \partial\mathbb{G}} \{V_P, 0\}$

Proof of DMP:

Assume not true:

$-\mathcal{L}_h V_p \le 0$ in $\mathbb{G}$, but there is $p^* \in \mathbb{G}$ s.t.

(*) $$V_{p^*} = \max_{\mathbb{G}} V_p > \max_{\partial\mathbb{G}} \{V_p, 0\} \quad (\ge 0)$$

Then

$$0 \ge -\mathcal{L}_h V_{p^*} = \alpha_{p^*p^*} V_{p^*} - \sum_{Q \ne p^*} \alpha_{p^*Q} V_Q$$

$$\overset{(A1)}{\ge} \sum_{Q \ne p^*} \alpha_{p^*Q} (V_{p^*} - V_Q)$$

$\alpha_{p^*Q} \ge 0$, V_{p^*} max

(**) $\Rightarrow V_Q = V_{p^*}$ when $\alpha_{p^*Q} > 0$.

By (A2) and (**), there is $P_1, \dots, P_k \in \mathbb{G}$, $Q \in \partial\mathbb{G}$ s.t.

$$V_{p^*} = V_{P_1} = V_{P_2} = \dots = V_{P_k} = V_Q \le \max_{P \in \partial\mathbb{G}} \{V_P, 0\}$$

This contradicts (*) $\Rightarrow$ Thm. 1 holds $\square$