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Exam solutions

1 **a)** Let $b(x) = \frac{2}{3} - x$, $P = (x_m, t_m)$, then (2) becomes

$$\frac{U_m^n - U_m^{n-1}}{k} + b(x_m)^+ \frac{U_m^{n-1} - U_{m-1}^{n-1}}{h} - b(x_m)^- \frac{U_{m+1}^{n-1} - U_m^{n-1}}{h} + 5U_m^{n-1} = 0,$$

for $m = 1, \ldots, M-1$, $n = 1, \ldots, N$, or multiplying by k and letting $r = \frac{k}{h}$,

$$U_m^n - \big(1 - r(\underbrace{b(x_m)^+ + b(x_m)^-}_{=|b(x_m)|}) - 5k\big)U_m^{n-1} - rb(x_m)^+ U_{m-1}^{n-1} - rb(x_m)^- U_{m+1}^{n-1} = 0,$$

or with $\alpha_m = 1$, $\alpha_{m,0} = 1 - r|b(x_m)| - 5k$, and $\alpha_{m,\pm 1} = rb(x_m)^{\pm}$,

$$\alpha_m U_m^n - \alpha_{m,0} U_m^{n-1} - \alpha_{m,-1} U_{m-1}^{n-1} - \alpha_{m,+1} U_{m+1}^{n-1} = 0.$$

Observe that

1. $\alpha_m,\ \alpha_{m,\pm 1} \geq 0$,
2. $\alpha_m - (\alpha_{m,0} + \alpha_{m,-1} + \alpha_{m,+1}) = 5k$,
3. $\alpha_{m,0} \geq 0 \iff r|b(x_m)| \leq 1 - 5k$.

Since (3) must hold for all m and k, the scheme will have positive coefficients when the following CFL condition holds:

$$\frac{k}{h}\|b\|_{L^\infty(0,1)} + 5k \leq 1 \iff k \leq \frac{h}{\|b\|_{L^\infty(0,1)} + 5h} \overset{b=\frac{2}{3}-x}{=} \frac{h}{\frac{2}{3} + 5h}.$$

b) By (1) and (2) and $a = a^+ - a^-$,

$$\begin{aligned}\tau_m^n &= Lu_m^n - L_h u_m^n \\ &= \Big(\partial_t u_m^n - \frac{\Delta_k}{k} u_m^{n-1}\Big) + a(x_m)^+ \Big(\partial_x u_m^n - \frac{\nabla_h}{h} u_m^{n-1}\Big) - a(x_m)^- \Big(\partial_x u_m^n - \frac{\Delta_h}{h} u_m^{n-1}\Big) \\ &\quad + 5(u_m^n - u_m^{n-1}).\end{aligned}$$

We Taylor expand to find that

$$\begin{aligned}u_{m\pm1}^{n-1} &= u_m^{n-1} \pm h(u_x)_m^{n-1} + \frac{h}{2} u_{xx}(\xi_m^{\pm}, t_{n-1}), \\ \frac{\Delta_h}{h} u_m^{n-1} &= \frac{u_{m+1}^{n-1} - u_m^{n-1}}{h} = (u_x)_m^{n-1} + \frac{1}{2} h u_{xx}(\xi_m^+, t_{n-1}), \\ \frac{\nabla_h}{h} u_m^{n-1} &= \frac{u_m^{n-1} - u_{m-1}^{n-1}}{h} = (u_x)_m^{n-1} - \frac{1}{2} h u_{xx}(\xi_m^-, t_{n-1}), \\ (u_x)_m^n &= (u_x)_m^{n-1} + k u_{xt}(x_m, \eta_n).\end{aligned}$$

For the difference in time, we expand around (x_m, t_n):

$$u_m^{n-1} = u_m^n - k(u_t)_m^n + \frac{1}{2}k^2 u_{tt}(x_m, \tilde{\eta}_n),$$
$$\frac{\Delta_k}{k}u_m^{n-1} = \frac{u_m^n - u_m^{n-1}}{k} = (u_t)_m^n - \frac{1}{2}k u_{tt}(x_m, \tilde{\eta}_n).$$

Combining these estiates, we find that

$$\tau_m^n = \frac{1}{2}k u_{tt}(x_m, \tilde{\eta}_n) + a(x_m)^+ \Big(\frac{1}{2}h u_{xx}(\xi_m^-, t_{n-1}) + k u_{xt}(x_m, \eta_n)\Big) + a(x_m)^- \Big(\frac{1}{2}h u_{xx}(\xi_m^+, t_{n-1}) - k u_{xt}(x_m, \eta_n)\Big) + 5k u_t(x_m, \bar{\eta}_n).$$

c) The $x(t)$-characteristic equation is $\dot{x} = b(x) = \frac{2}{3} - x$.

At the left boundary $x = 0$, $\dot{x} = b(0) = \frac{2}{3} > 0$
$\Rightarrow$ inflow, characteristics starting at the $x = 0$ go <u>into</u> $(0, 1)$

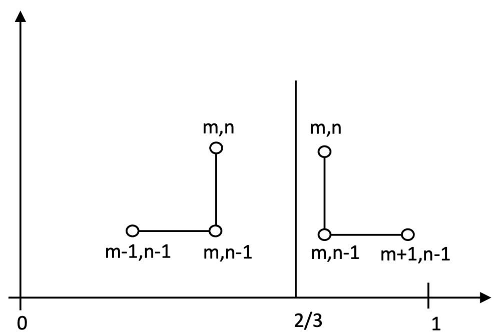
At the right boundary $x = 1$, $\dot{x} = b(1) = -\frac{1}{3} < 0$
$\Rightarrow$ inflow, characteristics starting at the $x = 1$ go <u>into</u> $(0, 1)$

Hence we must impose boundary conditions at both $x = 0$ and $x = 1$. Since initial conditions are needed at $t = 0$, we have

$$\partial^* Q_T = \{x = 0\} \times (0, T) \cup \{x = 1\} \times (0, T) \cup [0, 1] \times \{t = 0\}.$$

Stencils:

$$x_m < \frac{2}{3} \quad \Longrightarrow \quad b^- = 0 \text{ and stencil } (x_m, t_n), (x_{m-1}, t_{n-1}), (x_m, t_{n-1}),$$
$$x_m > \frac{2}{3} \quad \Longrightarrow \quad b^+ = 0 \text{ and stencil } (x_m, t_n), (x_m, t_{n-1}), (x_{m+1}, t_{n-1}).$$



It follows that starting from (x_m, t_n), the scheme will (iterating backwards) eventually reach $t = 0$ and $x = 0$ if $x_m < \frac{2}{3}$ and $x = 1$ if $x_m > \frac{2}{3}$. Hence the reachable boundary is

$$\partial^* \mathbb{G} = \partial^* Q_T \quad \text{(defined above)}.$$

d) Error $e_m^n = u_m^n - U_m^n$. By (1), (2), and the definition of τ,

$$L_h e_m^n = L_h u_m^n - L_h U_m^n = L u_m^n - \tau_m^n - 0 = 0 - \tau_m^n,$$

for interior points $(x_m, t_n) \in \mathbb{G}$. Since $\partial^*\mathbb{G} \subset \partial^* Q_T$ and u and U satisfy the same boundary and initial conditions,

$$e_m^n = 0 \quad \text{for} \quad (x_m, t_n) \in \partial^*\mathbb{G}.$$

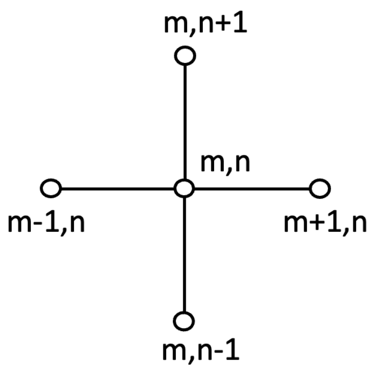
Hence by stability with respect to the right hand side, part b), and $\|a\|_{L^\infty} \leq \frac{2}{3}$,

$$\begin{aligned}
\max|u_m^n - U_m^n| &= \max|e_m^n| \leq \frac{1}{5}\max|\tau_m^n| \\
&\leq \frac{1}{5}\frac{1}{2}k\|u_{tt}\|_{L^\infty} + \frac{1}{5}\underbrace{\|a^+ + a^-\|}_{=\|a\|_{L^\infty}\leq\frac{2}{3}}\Big(\frac{1}{2}h\|u_{xx}\|_{L^\infty} + k\|u_{xt}\|_{L^\infty}\Big) + 5k\|u_t\|_{L^\infty} \\
&\leq \Big(\frac{1}{10}\|u_{tt}\|_{L^\infty} + \frac{2}{15}\|u_{xt}\|_{L^\infty} + 5\|u_t\|_{L^\infty}\Big)k + \frac{1}{15}h\|u_{xx}\|_{L^\infty}.
\end{aligned}$$

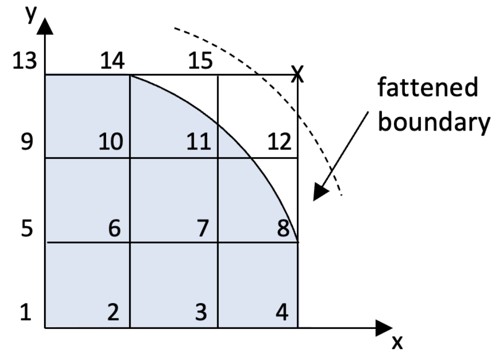
2 **a)** Central FDM approximation of L on $\bar{\mathbb{G}}_1$:

$$(*) \qquad L_h u_{m,n} = \frac{\delta_x^2}{h^2}u_{m,n} + (1 + x_m)\frac{\delta_y^2}{h^2}u_{m,n}.$$

Stencil:



Grid $\bar{\mathbb{G}}_1$:



Enumeration of $\bar{\mathbb{G}}_1$:

$$\begin{array}{llll} 1\ (x_0,y_0) & 2\ (x_1,y_0) & 3\ (x_2,y_0) & 4\ (x_3,y_0) \\ 5\ (x_0,y_1) & 6\ (x_1,y_1) & 7\ (x_2,y_1) & 8\ (x_3,y_1) \\ 9\ (x_0,y_2) & 10\ (x_1,y_2) & 11\ (x_2,y_2) & 12\ (x_3,y_2) \\ 13\ (x_0,y_3) & 14\ (x_1,y_3) & 15\ (x_2,y_3) & \times \end{array}$$

Interior nodes $\mathbb{G}_1$: 6,7,10,11. Boundary nodes $\partial\mathbb{G}_1$: 1-5,8,9,12-15.

Boundary conditions: Extend by constant from $\partial\Omega$ to $\partial\mathbb{G}_1$:

$$U(P) = u(P) = 0 \quad \text{for} \quad P \in \{P_1, \dots, P_5, P_9, P_{13}\},$$

$$U(P) = u(P) = 1 \quad \text{for} \quad P \in \{P_8, P_{14}\},$$

$$U(P) = 1 \quad \text{for} \quad P \in \{P_{12}, P_{15}\} \text{ by extension.}$$

b) The scheme can be written as

$$\begin{aligned} &-L_h U_{m,n} = 0 \\ &\quad \Updownarrow \quad (*)\cdot h^2 \\ &-(U_{m+1,n} - 2U_{m,n} + U_{m-1,n}) - (1+x_m)(U_{m,n-1} - 2U_{m,n} + U_{m,n+1}) = 0 \\ &\quad \Updownarrow \\ &2(2+x_m)U_{m,n} - U_{m+1,n} - U_{m-1,n} - (1+x_m)(U_{m,n-1} + U_{m,n+1}) = 0 \end{aligned}$$

The scheme at the interior nodes $P_6,\ P_7,\ P_{10},\ P_{11}$:

$$\begin{aligned} &U_6 = U_{1,1}: \\ &\quad 2(2+\frac{1}{3})U_6 - U_5 - U_7 - (1+\frac{1}{3})(U_2 + U_{10}) = 0 \\ &U_7 = U_{2,1}: \\ &\quad 2(2+\frac{2}{3})U_7 - U_6 - U_8 - (1+\frac{2}{3})(U_3 + U_{11}) = 0 \\ &U_{10} = U_{1,2}: \\ &\quad 2(2+\frac{1}{3})U_{10} - U_9 - U_{11} - (1+\frac{1}{3})(U_6 + U_{14}) = 0 \\ &U_{11} = U_{2,2}: \\ &\quad 2(2+\frac{2}{3})U_{11} - U_{10} - U_{12} - (1+\frac{2}{3})(U_7 + U_{15}) = 0 \end{aligned}$$

Using the boundary conditions from a), we then find that

$$\begin{bmatrix} \frac{14}{3} & -1 & -\frac{4}{3} & 0 \\ -1 & \frac{16}{3} & 0 & -\frac{5}{3} \\ -\frac{4}{3} & 0 & \frac{14}{3} & -1 \\ 0 & -\frac{5}{3} & -1 & \frac{16}{3} \end{bmatrix} \begin{bmatrix} U_6 \\ U_7 \\ U_{10} \\ U_{11} \end{bmatrix} = \begin{bmatrix} \frac{4}{3}U_2 + U_5 \\ \frac{5}{3}U_3 + U_8 \\ U_9 + \frac{4}{3}U_{14} \\ U_{12} + \frac{5}{3}U_{15} \end{bmatrix} \overset{B.C.}{=} \begin{bmatrix} 0 \\ 1 \\ \frac{4}{3} \\ \frac{8}{3} \end{bmatrix}.$$

3 **a)** To find the PDE and boundary conditions, we integrate by parts:

$$\begin{aligned} a(u,v) &= \int_0^1 (7u_x v_x - 5uv_x)\,\mathrm{d}x = \int_0^1 (7u_x - 5u)v_x\,\mathrm{d}x \\ &\overset{\text{i.b.p}}{=} -\int_0^1 (7u_x-5u)_x v\,\mathrm{d}x + \big[(7u_x-5u)v\big]_0^1 \\ &\overset{v(1)=0}{=} \int_0^1 (-7u_{xx}+5u_x)v\,\mathrm{d}x - (7u_x-5u)(0)v(0). \end{aligned}$$

Since $a(u,v) = F(v) = 3v(0)$, we should then have

$$-7u_{xx} + 5u_x = 0 \quad \text{in} \quad (0,1) \qquad \text{and} \qquad -(7u_x - 5u)(0) = 3.$$

Since $u \in V$, we also have that $u(1) = 0$.

a is continuous:

$$\begin{aligned} |a(u,v)| &\leq |\int_0^1 7u_x v_x| + |\int_0^1 5uv_x| \overset{C-S}{\leq} 7\|u_x\|_{L^2}\|v_x\|_{L^2} + 5\|u\|_{L^2}\|v_x\|_{L^2} \\ &\leq (7+5)\|u\|_{H^1}\|v\|_{H^1}. \end{aligned}$$

a is corecive:

$$\begin{aligned} a(u,u) &= \int_0^1 7u_x^2 - \int_0^1 5uu_x \overset{C-S}{\geq} 7\|u_x\|_{L^2}^2 - 5\|u\|_{L^2}\|u_x\|_{L^2} \\ &\overset{\|u\|_2 \leq \|u_x\|_2}{\geq} (7-5)\|u_x\|_{L^2}^2. \end{aligned}$$

Since $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u_x\|_{L^2}^2 \leq 2\|u_x\|_{L^2}^2$ ($\|u\|_{L^2} \leq \|u_x\|_{L^2}$), we have

$$a(u,u) \geq \frac{7-5}{2}\|u\|_{H^1}^2 = \|u\|_{H^1}^2.$$

b) Let $X_h^1 = \text{span}\{\varphi_0, \ldots, \varphi_m\}$ and $V_h = X_h^1 \cap V = \text{span}\{\varphi_0, \ldots, \varphi_{M-1}\}$. The P1-FEM for (4) is then

$$\text{Find } u_h \in V_h \text{ s.t. } a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$\Updownarrow \quad \varphi_0, \ldots, \varphi_{M-1} \text{ basis for } V_h$$

$$\text{Find } u_h \in V_h \text{ s.t. } a(u_h, \varphi_i) = F(\varphi_i), \quad i = 0, \ldots, M-1$$

$$\Updownarrow \quad u_h = \sum U_j \varphi_j(x) \text{ for some } U_j \in \mathbb{R}, \quad a \text{ bilinear}$$

$$\sum_{j=1}^{M-1} a(\varphi_j, \varphi_i) U_j = F(\varphi_i), \quad i = 0, \ldots, M-1$$

$$\Updownarrow$$

$$A\vec{U} = \vec{F}, \qquad \text{where} \quad A_{ij} = a(\varphi_j, \varphi_i) \quad \text{and} \quad F_i = F(\varphi_i)$$

We observe that $\varphi_i \neq 0$ only on (x_{i-1}, x_{i+1}), $\varphi_0 \neq 0$ only on (x_0, x_1),

$$\varphi_i' = \begin{cases} \frac{1}{h}, & \text{on } (x_{i-1}, x_i), \\ -\frac{1}{h}, & \text{on } (x_i, x_{i+1}), \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \int_{x_{i-1}}^{x_i} \varphi_i\,\mathrm{d}x = \int_{x_i}^{x_{i+1}} \varphi_i\,\mathrm{d}x = \frac{1}{2} \cdot h \cdot 1.$$

For $i = 1, \dots, M-1$ we then compute

$$\begin{aligned}
A_{ii} = a(\varphi_i, \varphi_i) &= \int_0^1 (\dots)\,\mathrm{d}x = \sum_{j=1}^{M} \int_{x_{j-1}}^{x_j} (\dots)\,\mathrm{d}x \\
&= 0 + \int_{x_{i-1}}^{x_i} (7\varphi_i' \cdot \varphi_i' - 5\varphi_i\varphi_i')\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} (7\varphi_i' \cdot \varphi_i' - 5\varphi_i\varphi_i')\,\mathrm{d}x \\
&= \int_{x_{i-1}}^{x_i} \Big(7\frac{1}{h}\cdot\frac{1}{h} - 5\frac{1}{h}\varphi_i\Big)\mathrm{d}x + \int_{x_i}^{x_{i+1}} \Big(7(-\frac{1}{h})\cdot(-\frac{1}{h}) - 5(-\frac{1}{h})\varphi_i\Big)\,\mathrm{d}x \\
&= 7\frac{1}{h^2}h - \frac{5}{h}\int_{x_{i-1}}^{x_i} \varphi_i\,\mathrm{d}x + 7\frac{1}{h^2}h + \frac{5}{h}\int_{x_i}^{x_{i+1}} \varphi_i\,\mathrm{d}x = \frac{14}{h} + 0,
\end{aligned}$$

$$\begin{aligned}
A_{i,i+1} = a(\varphi_{i+1}, \varphi_i) &= 0 + \int_{x_i}^{x_{i+1}} \Big(7\varphi_{i+1}'\varphi_i' - 5\varphi_{i+1}\varphi_i'\Big)\mathrm{d}x \\
&= 7\frac{1}{h}(-\frac{1}{h})h - 5\frac{1}{h}\int_{x_i}^{x_{i+1}} \varphi_{i+1}\mathrm{d}x = -\frac{7}{h} - \frac{5}{2},
\end{aligned}$$

$$A_{i-1,i} = a(\varphi_i, \varphi_{i-1}) = \int_{x_{i-1}}^{x_i} \Big(7\varphi_i'\varphi_{i-1}' - 5\varphi_i\varphi_{i-1}'\Big)\mathrm{d}x = -\frac{7}{h} + \frac{5}{2}.$$

For $i = 0$ we find that

$$\begin{aligned}
A_{00} = a(\varphi_0, \varphi_0) &= \int_{x_0}^{x_1} \Big(7\varphi_0'\varphi_0' - 5\varphi_0\varphi_0'\Big)\mathrm{d}x = \frac{7}{h} + \frac{5}{2}, \\
A_{01} = a(\varphi_1, \varphi_0) &= \int_{x_0}^{x_1} \Big(7\varphi_1'\varphi_0' - 5\varphi_0\varphi_0'\Big)\mathrm{d}x = -\frac{7}{h} + \frac{5}{2}.
\end{aligned}$$

For all other indices $\varphi_i\varphi_j = 0$ on $[0, 1]$, and hence $A_{i,j} = 0$.

For the right hand side we compute

$$F_i = F(\varphi_i) = 3\phi_i(0) = \begin{cases} 0, & i \neq 0, \\ 3, & i = 0. \end{cases}$$