

Problem 1 Consider the elliptic boundary value problem on $(0, 1)$,

$$(1) \quad \begin{cases} -Lu := -\frac{d}{dx} \left((1+x) \frac{du}{dx} \right) + 2u = 4x^3 & \text{in } (0, 1), \\ u(0) = \sqrt{2}, \quad u(1) = \sqrt{3}. \end{cases}$$

Note that L is self-adjoint. Let $M \in \mathbb{N}$, $h = \frac{1}{M}$, and $\overline{\mathbb{G}}$ be a uniform grid on $[0, 1]$,

$$\overline{\mathbb{G}} = \{x_m = mh : m = 0, \dots, M\}.$$

- a) Propose a self-adjoint (symmetric) 2nd order finite difference discretization of (1) on the grid $\overline{\mathbb{G}}$.

Determine the matrix A and vector \vec{F} in the corresponding linear system

$$A \vec{U} = \vec{F} \quad \text{where} \quad \vec{U} = (U_1, \dots, U_{M-1})^T.$$

Hint: You do *not* need to show that the scheme is self-adjoint or 2nd order.

Let L_h denote the approximation of L you proposed in part a).

- b) Show that $-L_h$ has positive coefficients.

Prove in all details that $-L_h$ satisfies the discrete maximum principle.

- c) Show that the approximation is max-norm stable in the sense that for any function $f : [0, 1] \rightarrow \mathbb{R}$,

$$\begin{cases} -L_h U_m = f(x_m), & m = 1, \dots, M-1, \\ U_0 = 0 = U_M. \end{cases} \quad \implies \quad \max_{m=0, \dots, M} |U_m| \leq \frac{1}{2} \max_{m=0, \dots, M} |f(x_m)|.$$

Problem 2 Consider the transport equation on $(0, 1)$,

$$(2) \quad u_t + b(x)u_x = f(x) \quad \text{in} \quad (0, 1) \times (0, T).$$

a) Explain where Dirichlet boundary conditions need to be imposed when

- (i) $b(x) = x - 3$,
- (ii) $b(x) = 1 - 2x$,
- (iii) $b(x) = 2x - 1$,
- (iv) $b(x) = (2x - 1)^2 - \frac{1}{2}$.

Let $M \in \mathbb{N}$, $x_m = mh$, and $h = \frac{1}{M}$ and $k > 0$ denote time and space steps. A Wendroff scheme for a transport equation is given by

$$U_{m+1}^{n+1} + \gamma_m U_m^{n+1} = \gamma_m U_{m+1}^n + U_m^n, \quad \gamma_m = \frac{2h - \pi k}{2h + \pi k}.$$

b) Use Taylor expansions to

- (i) identify which transport equation is being approximated, and
- (ii) show that the local truncation error is $O(h^2 + k^2)$ when computed on the solution u of this transport equation.

Hint: Use Taylor expansions! You get no points for repeating the derivation of the scheme.

Problem 3 Let $M \in \mathbb{N}$, $x_i = \frac{i}{M}$ for $i = 0, 1, \dots, M$, and $\vec{U} = (U_1, \dots, U_{M-1})$ be the solution of the finite difference equation

$$(3) \quad -\frac{\delta_h^2 U_i}{h^2} + (1 + x_i)^2 U_i = \sin x_i \quad i = 1, \dots, M-1; \quad U_0 = 0 = U_M.$$

Show that the k -th step $\vec{U}^{(k)}$ in the Jacobi iteration to solve equation (3) satisfies

$$\|\vec{U} - \vec{U}^{(k)}\|_\infty \leq \left(\frac{1}{1 + \frac{1}{2M^2}} \right)^k \|\vec{U} - \vec{U}^{(0)}\|_\infty \quad \text{for } k = 1, 2, \dots$$

Problem 4

a) Show that the bilinear form

$$a(u, v) = \iint_{[0,1]^2} \left((1+x^2) \nabla u \cdot \nabla v + (1-y^2) uv \right) dx dy$$

is continuous and coercive on $H_0^1([0, 1]^2)$.

Hint: You may use that $\|u\|_{L^2([0,1]^2)} \leq 2\|\nabla u\|_{L^2([0,1]^2)}$ for $u \in H_0^1([0, 1]^2)$.

Consider the one-dimensional elliptic boundary value problem

$$(4) \quad \begin{cases} -Lu := -\frac{d}{dx} \left((1+x) \frac{du}{dx} \right) + 2u = 2x & \text{in } (0, 1), \\ u(0) = \sqrt{2}, \quad u(1) = \sqrt{3}. \end{cases}$$

Let $M \in \mathbb{N}$ and $\mathcal{T}_h = \cup_{i=1}^M K_i$ be a triangulation of $(0, 1)$ with $K_i = (x_{i-1}, x_i)$, $x_i = ih$, and $h = \frac{1}{M}$.

b) Compute the elemental matrix A^{K_i} and elemental load vector \vec{F}^{K_i} for the \mathbb{P}_1 Lagrange finite element method for (4) on the triangulation \mathcal{T}_h .

Hint: Local basis. The use of symmetry and a change of variables can drastically simplify computations.