TMA4212 Numdiff
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Final Exam solution

Norwegian University of Science and Technology
Department of Mathematical Sciences

1 **a)** Let $a(x) = 1 + x$, $f(x) = 4x^3$, then a symmetric second order FDM approximation of (1) is given by

$$-\frac{\delta_h}{h}\left(a_m \frac{\delta_h}{h} U_m\right) + 2U_m = f_m,$$
$$\Updownarrow \cdot h^2$$
$$-a_{m+\frac{1}{2}}\left(U_{m+1} - U_m\right) + a_{m-\frac{1}{2}}\left(U_m - U_{m-1}\right) + 2h^2 U_m = h^2 f_m,$$
$$\Updownarrow$$
$$-a_{m+\frac{1}{2}} U_{m+1} + \left(a_{m+\frac{1}{2}} + a_{m-\frac{1}{2}} + 2h^2\right) U_m - a_{m-\frac{1}{2}} U_{m-1} = h^2 f_m,$$

for $m = 1, \ldots, M-1$. Using the boundary conditions, we find that for $m = 1$ and $m = M-1$,

$$\begin{aligned} -a_{\frac{3}{2}} U_2 + \left(a_{\frac{3}{2}} + a_{\frac{1}{2}} + 2h^2\right) U_1 &= h^2 f_1 + a_{\frac{1}{2}} U_0 \\ &= h^2 f_1 + a_{\frac{1}{2}} \sqrt{2}, \\ \left(a_{M-\frac{1}{2}} + a_{M-\frac{3}{2}} + 2h^2\right) U_{M-1} - a_{M-\frac{3}{2}} U_{M-2} &= h^2 f_m + a_{M-\frac{1}{2}} U_M \\ &= h^2 f_{M-1} + a_{M-\frac{1}{2}} \sqrt{3}. \end{aligned}$$

We conclude that $\vec{U} = [U_1, \ldots, U_{M-1}]^T$ solves the linear system $A\vec{U} = \vec{F}$ with

$$\begin{aligned} A &= \text{tridiag}\left\{-a_{m-\frac{1}{2}}, \left(a_{m+\frac{1}{2}} + a_{m+\frac{1}{2}} + 2h^2\right), -a_{m+\frac{1}{2}}\right\}, \\ \vec{F} &= h^2 \vec{f} + [a_{\frac{1}{2}}\sqrt{2}, 0, \ldots, 0, a_{M-\frac{1}{2}}\sqrt{3}]^T. \end{aligned}$$

b) Observe that

$$\begin{aligned} -h^2 L_h U_m &= -a_{m-\frac{1}{2}} U_{m-1} + \left(a_{m+\frac{1}{2}} + a_{m-\frac{1}{2}} + 2h^2\right) U_m - a_{m+\frac{1}{2}} U_{m+1} \\ &= \alpha_{m,m} U_m - \sum_{k \neq m} \alpha_{m,k} U_k, \end{aligned}$$

with $\alpha_{m,m} = a_{m+\frac{1}{2}} + a_{m-\frac{1}{2}} + 2h^2$, $\alpha_{m,m\pm 1} = a_{m\pm\frac{1}{2}}$, and $\alpha_{m,k} = 0$ for $k \neq m, m \pm 1$. For every m, n

$$\alpha_{mn} \geq 0 \qquad \text{and} \qquad \sum_{k \neq m} \alpha_{mk} = a_{m-\frac{1}{2}} + a_{m+\frac{1}{2}} < \alpha_{m,m},$$

and it follows that $-L_h$ has *positive coefficients*.

The discrete maximum principle (DMP):

$$(1)\quad -L_h U_m \le 0, \quad m = 1, \ldots, M-1 \quad \Longrightarrow \quad \max_m U_m \le \max\{0, U_0, U_m\}.$$

Proof. Assume

$$(2)\qquad \max_m U_m = U_{\bar{m}} > 0 \qquad \text{for some} \qquad \bar{m} \in \{1, \ldots, M-1\}.$$

Then

$$\begin{aligned}
0 \ge -h^2 L_h U_{\bar{m}} &= \alpha_{\bar{m},\bar{m}} U_{\bar{m}} - \alpha_{\bar{m},\bar{m}+1} U_{\bar{m}+1} - \alpha_{\bar{m},\bar{m}-1} U_{\bar{m}-1} \\
&= \underbrace{(\alpha_{\bar{m},\bar{m}} - \alpha_{\bar{m},\bar{m}+1} - \alpha_{\bar{m},\bar{m}-1})}_{2h^2} \underbrace{U_{\bar{m}}}_{>0} \\
&\quad + \underbrace{\alpha_{\bar{m},\bar{m}+1}}_{\ge 0} \underbrace{(U_{\bar{m}} - U_{\bar{m}+1})}_{\ge 0} + \underbrace{\alpha_{\bar{m},\bar{m}-1}}_{\ge 0} \underbrace{(U_{\bar{m}} - U_{\bar{m}-1})}_{\ge 0} \\
&\ge 2h^2 U_{\bar{m}} > 0,
\end{aligned}$$

which contradicts $-L_h U_m \le 0,\ m = 1, \ldots, M-1$. Hence (2) cannot hold, and then either $U_{\bar{m}} \le 0$ or $\bar{m} \in \{0, M\}$. In either case $\max_m U_m \le \max\{0, U_0, U_m\}$ and (1) holds. □

c) Let $V_m := U_m - \|f\|_{L^\infty} \phi_m$, where $\phi_m = \frac{1}{2}$ is a supersolution of $-L_h$ satifying $-L_h \phi_m = 1$. Then

$$-L_h V_m = -L_h U_m - \|f\|_{L^\infty} (-L_h)\phi_m = f_m - \|f\|_{L^\infty} \le 0,\ m = 1, \ldots, M-1,$$

and DMP and $V_m \le U_m$ imply that

$$V_m \le \max\{0, V_0, V_M\} \le \max\{0, U_0, U_M\} \quad \forall m.$$

When $U_0 = 0 = U_M$, we find that $V_m \le 0$ and

$$U_m = V_m + \phi_m \|f\|_{L^\infty} \le 0 + \frac{1}{2}\|f\|_{L^\infty} \quad \forall m.$$

Repeat for $-L_h(-U_m) = -f_m$, $m = 1, \ldots, M-1$, to get $-U_m \le \frac{1}{2}\|-f\|_{L^\infty}$, and conclude that

$$\max_m |U_m| \le \frac{1}{2}\|f\|_{L^\infty}.$$

$\boxed{2}$ **a)** Here the x-characteristic equation is $\dot{x} = b(x)$. Boundary conditions for (2) must (and can only) be imposed at inflow, where the x-charateristic curves go into the domain $(0,1)$. So at $x = 0$ if $b(0) > 0$ and $x = 1$ if $b(1) < 0$.

case	$b(0)$	$b(1)$	BC at $x = 0$	BC at $x = 1$
(i)	< 0	< 0	No	Yes
(ii)	> 0	< 0	Yes	Yes
(iii)	< 0	> 0	No	No
(iv)	> 0	> 0	Yes	No

b) Taylor expand about $u = u_m^n$:

$$\begin{aligned}
u_{m+1}^{n+1} &= u + hu_x + ku_t + \frac{1}{2}h^2 u_{xx} + khu_{xt} + \frac{1}{2}k^2 u_{tt} + \mathcal{O}(h^3 + k^3),\\
u_m^{n+1} &= u + ku_t + \frac{1}{2}k^2 u_{tt} + \mathcal{O}(k^3),\\
u_{m+1}^{n} &= u + hu_x + \frac{1}{2}h^2 u_{xx} + \mathcal{O}(h^3).
\end{aligned}$$

Note that $\gamma_m = \frac{2-\frac{\pi k}{h}}{2+\frac{\pi k}{h}} = \frac{1-r}{1+r}$ for $r = \frac{\pi k}{2h}$, and multiply the scheme by (1+r),

$$(1+r)u_{m+1}^{n+1} + (1-r)u_m^{n+1} = (1-r)u_{m+1}^n + (1+r)u_m^{n+1},$$

use the Taylor expansions, and collect terms:

$$\begin{aligned}
&((1+r) + (1-r) - (1-r) - (1+r))\, u\\
&+ ((1+r) - (1-r))\, hu_x\\
&+ ((1+r) + (1-r))\, ku_t\\
&+ ((1+r) - (1-r))\, \frac{1}{2}h^2 u_{xx}\\
&+ ((1+r) + (1-r))\, \frac{1}{2}k^2 u_{tt}\\
&+ (1+r)hku_{xt}\\
&+ \mathcal{O}(h^3 + k^3) = 0.
\end{aligned}$$

Since $2hr = \pi k$, we get

$$0 \cdot u + \pi k u_x + 2ku_t + \frac{1}{2}\pi hk u_{xx} + k^2 u_{tt} + (kh + \frac{1}{2}\pi k^2)u_{xt} + \mathcal{O}(h^3 + k^3) = 0,$$

and multiplication by $\frac{1}{2k}$ gives

$$(*) \qquad u_t + \frac{\pi}{2}u_x + \frac{1}{2}h\left(\frac{\pi}{2}u_{xx} + u_{xt}\right) + \frac{1}{2}k\left(u_{tt} + \frac{\pi}{2}u_{xt}\right) = \mathcal{O}(h^3 + k^3).$$

Hence $u_t + \frac{\pi}{2}u_x = \mathcal{O}(h + k)$, and

$$(**) \qquad u_t + \frac{\pi}{2}u_x = 0$$

is the transport equation being approximated.

To continue, note that by $(**)$

$$u_{tt} = \left(-\frac{\pi}{2}u_x\right)_t = -\frac{\pi}{2}u_{xt} \qquad \text{and} \qquad u_{xx} = \left(-\frac{2}{\pi}u_t\right)_x = -\frac{2}{\pi}u_{xt},$$

and hence $(*)$ implies

$$u_t + \frac{\pi}{2}u_x + 0 + 0 = \mathcal{O}(h^3 + k^3).$$

The local truncation error is then

$$\begin{aligned}
\tau_m &= -Lu_m - (-L_h u_m)\\
&= u_t + \frac{\pi}{2}u_x - \left(u_{m+1}^{n+1} + \gamma_m u_m^{n+1} - \gamma_m u_{m+1}^n + u_m^n\right)\frac{1+r}{2k}\\
&= \mathcal{O}(h^3 + k^3).
\end{aligned}$$

$\boxed{3}$ Multiplying scheme (3) by h^2 gives

$$-U_{i-1} + \left(2 + h^2(1+x_i)^2\right) U_i - U_{i+1} = h^2 \sin x_i, \quad i = 1, \dots, M-1,$$
$$U_0 = 0 = U_M,$$

which is equivalent to the linear system $A\vec{U} = \vec{b}$ with

$$A = \text{tridiag}\{-1, 2 + h^2(1+x_i)^2, -1\} \qquad \text{and} \qquad b_i = h^2 \sin x_i,\ i = 1, \dots, M-1.$$

We split the matrix A, $A = -L + D - U$, where

$$L = \text{tridiag}\{1, 0, 0\}, \quad U = \text{tridiag}\{0, 0, 1\}, \quad D = \text{diag}\{\underbrace{2 + h^2(1+x_i)^2}_{=:d(x_i)}\},$$

and define the Jacobi-iteration as

$$D\vec{U}^{(k+1)} = (L+U)\vec{U}^{(k)} + \vec{b} \qquad \text{or} \qquad \vec{U}^{(k+1)} = B\,\vec{U}^{(k)} + D^{-1}\vec{b}$$

for the iteration matrix $B = D^{-1}(L+U) = \text{tridiag}\left\{\frac{1}{d(x_i)}, 0, \frac{1}{d(x_i)}\right\}$. Since

$$A\vec{U} = \vec{b} \quad \iff \quad D\vec{U} = (L+U)\vec{U} + \vec{b} \quad \iff \quad \vec{U} = B\vec{U} + D^{-1}\vec{b},$$

the iteration error satisfies

$$\vec{U} - \vec{U}^{(k)} = B\,(\vec{U} - \vec{U}^{(k-1)}) = \cdots = B^k\,(\vec{U} - \vec{U}^{(0)})$$
$$\implies \quad \left\|\vec{U} - \vec{U}^{(k)}\right\|_\infty \le \left\|B^k\right\|_\infty \left\|\vec{U} - \vec{U}^{(0)}\right\|_\infty.$$

To estimate $\|B^k\|_\infty$, we use propeperties of subordnate norms, the max row-sum characterisation of the subordinate max-norm, and $h = \frac{1}{M}$,

$$\|B^k\|_\infty \le \|B\|_\infty^k,$$
$$\|B\|_\infty = \max_j \sum_i |B_{ij}| = \max_j 2\frac{1}{d(x_j)}$$
$$\le 2\frac{1}{\min d(x_j)} = \frac{2}{d(x_1)} = \frac{2}{2 + \frac{1}{M^2}(1+\frac{1}{M})^2} \le \frac{1}{1 + \frac{1}{2M^2}}.$$

Conclusion:

$$\|\vec{U} - \vec{U}^{(k)}\|_\infty \le \left(\frac{1}{1 + \frac{1}{2M^2}}\right)^k \|\vec{U} - \vec{U}^{(0)}\|_\infty, \quad k = 1, 2, \dots$$

$\boxed{4}$ **a)** *a continuous on H^1:* Take absolute value inside integral, use $|1 + x^2| \le 2$ and $|1 - y^2| \le 1$ on $[0,1]^2$, the Cauchy-Schwartz inequality, and $\|u\|_{H^1}^2 = \|u\|_{L^2} + \|\nabla u\|_{L^2}$,

$$|a(u,v)| \le \int_0^1 \int_0^1 (1+x^2)|\nabla u||\nabla v| \,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^1 (1-y^2)|u||v| \,\mathrm{d}x\,\mathrm{d}y$$
$$\le 2\int_0^1 \int_0^1 |\nabla u||\nabla v| \,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^1 |u||v| \,\mathrm{d}x\,\mathrm{d}y$$
$$\le 2\,\|\nabla u\|_{L^2}\,\|\nabla v\|_{L^2} + \|u\|_{L^2}\,\|v\|_{L^2} \le 2\,\|u\|_{H^1}\,\|v\|_{H^1}.$$

a coercive on H^1: We use that $|1+x^2| \geq 1$ and $|1-y^2| \geq 0$ on $[0,1]^2$ along with the Poincare inequality given in the problem,

$$
\begin{aligned}
a(u,v) &= \int_0^1 \int_0^1 (1+x^2)|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_0^1 \int_0^1 (1-y^2)u^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\geq \int_0^1 \int_0^1 |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y + 0 \geq \frac{1}{5} \|u\|_{H^1}^2 .
\end{aligned}
$$

The last inequality follows from the Poincaré inequality:

$$
\|u\|_{L^2} \leq 2\|\nabla u\|_{L^2} \implies \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq 5\|\nabla u\|_{L^2}^2 .
$$

b) The local basis on element $K_i = (x_{i-1}, x_i)$ is

$$
\phi_0(x) = 1 - \tfrac{x_{i-1}-x}{h}, \qquad \phi_1(x) = \tfrac{x_{i-1}-x}{h}.
$$

Using the change of variable $x = x_{i-1} + hy$, $dx = h\,dy$, we find that

$$
\begin{aligned}
a^{K_i}(\phi_0,\phi_0) &= \int_{x_{i-1}}^{x_i} (1+x)(\phi_0)_x^2 + 2\phi_0^2 \,\mathrm{d}x \\
&= \int_0^1 \left((1+x_{i-1}+hy)\frac{1}{h^2} + 2(1-y)^2 \right) h \,\mathrm{d}y \\
&= \frac{1}{h^2}\left[(1+x_{i-1})y + \frac{1}{2}hy^2 \right]_0^1 h + 2\left[\frac{1}{3}y^3\right]_0^1 h \qquad \left[\textstyle\int_0^1 (1-y)^2 dy = \int_0^1 z^2 dz \right] \\
&= \frac{1}{h}\left(1 + x_{i-\frac{1}{2}}\right) + \frac{2}{3}h,
\end{aligned}
$$

$$
\begin{aligned}
a^{K_i}(\phi_0,\phi_1) &= \int_{x_{i-1}}^{x_i} (1+x)(\phi_0)_x(\phi_1)_x + 2\phi_0\phi_1 \,\mathrm{d}x \\
&= -\int_{x_{i-1}}^{x_i} (1+x)(\phi_0)_x^2 \,\mathrm{d}x + 2h\int_0^1 (1-y)y \,\mathrm{d}y \\
&= -\frac{1}{h}\left(1 + x_{i-\frac{1}{2}}\right) + 2h\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 \\
&= -\frac{1}{h}\left(1 + x_{i-\frac{1}{2}}\right) + \frac{2}{6}h,
\end{aligned}
$$

By symmetry, $a^{K_i}(\phi_0,\phi_0) = a^{K_i}(\phi_1,\phi_1)$ and $a^{K_i}(\phi_0,\phi_1) = a^{K_i}(\phi_1,\phi_0)$.

$$
\begin{aligned}
F^{K_i}(\phi_0) &= \int_{x_{i-1}}^{x_i} 2x\Big(1 - \frac{x - x_{i-1}}{h}\Big) \,\mathrm{d}x = \int_0^1 2\,(x_{i-1}+hy)\,(1-y)\,h \,\mathrm{d}y \\
&= \Big(2(x_{i-1} + \tfrac{1}{2}h) - 2(\tfrac{1}{2}x_{i-1} + \tfrac{1}{3}h)\Big)h = hx_{i-1} + \tfrac{1}{3}h^2,
\end{aligned}
$$

$$
F^{K_i}(\phi_1) = \int_{x_{i-1}}^{x_i} 2x\,\frac{x - x_{i-1}}{h} \,\mathrm{d}x = \int_0^1 2(x_{i-1}+hy)yh \,\mathrm{d}y = hx_{i-1} + \frac{2}{3}h^2.
$$

The elemental matrix and load vector is then given by

$$
A^{K_i} = \left(1 + x_{i-\frac{1}{2}}\right)\frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{3}h\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},
$$

$$
F^{K_i} = hx_{i-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}h^2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.
$$