Orthogonal polynomials.

The aim of this section is to construct "optimal" quadrature formulas. To be more specific, given the integral

$$I_w(f) = \int_a^b w(x)f(x)dx \tag{1}$$

in which $w(x)$ is a fixed, positive function. We want to approximate this using a quadrature formula on the form

$$Q_w(f) = \sum_{i=0}^{n} A_i f(x_i).$$

Such a formula can be constructed as follows: Choose $n+1$ distinct nodes, $x_0, x_1, \cdots, x_n$ in the interval $[a, b]$. Construct the interpolation polynomial

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x), \qquad \ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

An approximation to the integral is then given by

$$Q_w(f) = \int_a^b w(x)p_{n-1}(x)dx = \sum_{i=0}^{m} A_i f(x_i), \qquad A_i = \int_a^b w(x)\ell_i(x)dx. \tag{2}$$

The quadrature formula is of precision m if

$$I_w(p) = Q_w(p), \qquad \text{for all } p \in \mathbb{P}_m.$$

From the construction, these quadrature formulas is of precision at least n. The question is how too choose the nodes x_i, $i = 0, \ldots, n$ giving m as large as possible. The key concept here is *orthogonal polynomials.*

Orthogonal polynomials.

Given two functions $f, g \in C[a, b]$. We define an inner product of these two functions by

$$< f, g >= \int_a^b w(x)f(x)g(x)dx, \qquad w(x) > 0. \tag{3}$$

Thus the definition of the inner product depends on the integration interval $[a, b]$ and a given *weight function* $w(x)$. If $f, g, h \in C[a, b]$ and $\alpha \in \mathbb{R}$ then

$$\begin{aligned}
\langle f, g\rangle_w &= \langle g, f\rangle_w \\
\langle f + g, h\rangle_w &= \langle f, h\rangle_w + \langle g, h\rangle_w \\
\langle \alpha f, g\rangle_w &= \alpha\langle f, g\rangle_w \\
\langle f, f\rangle_w &\geq 0, \qquad \text{and} \qquad \langle f, f\rangle_w = 0 \Leftrightarrow f \equiv 0.
\end{aligned}$$

From an inner product, we can also define a norm on $C[a, b]$ by

$$\|f\|_w^2 = \langle f, f\rangle_w.$$

For the inner product (3) we also have

$$\langle xf, g\rangle_w = \int_a^b w(x)xf(x)g(x)dx = \langle f, xg\rangle_w. \tag{4}$$

Our aim is now to create an orthogonal basis for $\mathbb{P}$, that is, create a sequence of polynomials $\phi_k(x)$ of degree k (no more, no less) for $k = 0, 1, 2, 3, \dots$ such that

$$\langle \phi_i, \phi_j\rangle_w = 0 \quad \text{for all} \quad i \neq j.$$

If we can make such a sequence, then

$$\mathbb{P}_{n-1} = \text{span}\{\phi_0, \phi_1, \cdots, \phi_{n-1}\} \qquad \text{and} \qquad \langle \phi_n, p\rangle_w = 0 \quad \text{for all} \qquad p \in \mathbb{P}_{n-1}.$$

Let us now find the sequence of orthogonal polynomials. This is done by a Gram-Schmidt process:

Let $\phi_0 = 1$. Let $\phi_1 = x - B_1$ where B_1 is given by the orthogonality condition:

$$0 = \langle \phi_1, \phi_0\rangle_w = \langle x, 1\rangle_w - B_1\langle 1, 1\rangle_w \qquad \Rightarrow \qquad B_1 = \frac{\langle x, 1\rangle_w}{\|1\|_w^2}.$$

Let us now assume that we have found ϕ_j, $j = 0, 1, \dots, k-1$. Then, let

$$\phi_k = x\phi_{k-1} - \sum_{j=0}^{k-1} \alpha_j \phi_j.$$

Clearly, ϕ_k is a polynomial of degree k, and α_j can be chosen so that $\langle \phi_k, \phi_i\rangle_w = 0$, $i = 0, 1, \dots, k-1$, or

$$\langle \phi_k, \phi_i\rangle_w = \langle x\phi_{k-1}, \phi_i\rangle_w - \sum_{j=0}^{k-1} \alpha_j \langle \phi_i, \phi_j\rangle_w = \langle x\phi_{k-1}, \phi_i\rangle_w - \alpha_i \langle \phi_i, \phi_i\rangle_w = 0, \qquad i = 0, 1, \cdots, k-1.$$

So $\alpha_i = \langle x\phi_{k-1}, \phi_i\rangle_w / \langle \phi_i, \phi_i\rangle_w$. But we can do even better. Since ϕ_{k-1} is orthogonal to all polynomials of degree $k-2$ or less, we get

$$\langle x\phi_{k-1}, \phi_i\rangle_w = \langle \phi_{k-1}, x\phi_i\rangle_w = 0 \quad \text{for} \quad i+1 < k-1.$$

So, we are left only with α_{k-1} and α_{k-2}. The following theorem concludes the argument:

Theorem 1. *The sequence of orthogonal polynomials can be defined as follows:*

$$\begin{aligned} &\phi_0(x) = 1, \qquad \phi_1(x) = x - B_1 \\ &\phi_k(x) = (x - B_k)\phi_{k-1}(x) - C_k\phi_{k-2}(x), \qquad k \geq 2 \end{aligned}$$

with

$$B_k = \frac{\langle x\phi_{k-1}, \phi_{k-1}\rangle_w}{\|\phi_{k-1}\|_w^2}, \qquad C_k = \frac{\langle x\phi_{k-1}, \phi_{k-2}\rangle_w}{\|\phi_{k-2}\|_w^2} = \frac{\|\phi_{k-1}\|_w^2}{\|\phi_{k-2}\|_w^2}$$

The last simplification of C_k is given by:

$$\langle x\phi_{k-1}, \phi_{k-2}\rangle_w = \langle \phi_{k-1}, x\phi_{k-2}\rangle_w$$
$$\phi_{k-1} = x\phi_{k-2} - B_{k-1}\phi_{k-2} - C_{k-1}\phi_{k-3}.$$

Solve the second with respect to $x\phi_{k-2}$, replace it into the right hand side of the first expression, and use the orthogonality conditions.

Example 2. *For the inner product*

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$$

we get

$$\begin{array}{lllll}
\phi_0 = 1, & \langle x\phi_0, \phi_0\rangle = 0, & \langle \phi_0, \phi_0\rangle = 2, & B_1 = 0, & \\
\phi_1 = x, & \langle x\phi_1, \phi_1\rangle = 0, & \langle \phi_1, \phi_1\rangle = \frac{2}{3}, & B_2 = 0, & C_2 = \frac{1}{3} \\
\phi_2 = x^2 - \frac{1}{3} & \langle x\phi_2, \phi_2\rangle = 0, & \langle \phi_2, \phi_2\rangle = \frac{8}{45}, & B_3 = 0, & C_3 = \frac{4}{15} \\
\phi_3 = x^3 - \frac{3}{5}x, & \textit{etc.} & & &
\end{array}$$

These are the well known Legendre polynomials.

Example 3. *Let $w(x) = 1/\sqrt{1-x^2}$, and $[a, b] = [-1, 1]$. We then get the sequence of polynomials:*

$$\begin{array}{lllll}
\phi_0 = 1, & \langle x\phi_0, \phi_0\rangle_w = 0, & \langle \phi_0, \phi_0\rangle_w = \pi, & B_1 = 0, & \\
\phi_1 = x, & \langle x\phi_1, \phi_1\rangle_w = 0, & \langle \phi_1, \phi_1\rangle_w = \frac{\pi}{2}, & B_2 = 0, & C_2 = \frac{1}{2} \\
\phi_2 = x^2 - \frac{1}{2} & \langle x\phi_2, \phi_2\rangle_w = 0, & \langle \phi_2, \phi_2\rangle_w = \frac{\pi}{2}, & B_3 = 0, & C_3 = \frac{1}{4} \\
\phi_3 = x^3 - \frac{3}{4}x, & \textit{etc.} & & &
\end{array}$$

These are nothing but the monic Chebyshev polynomials $\tilde{T}_k$.

The following theorem will become useful:

Theorem 4. *Let $f \in C[a, b]$, $f \not\equiv 0$ satisfying $\langle f, p\rangle_w = 0$ for all $p \in P_{k-1}$. Then f changes signs at least k times on (a, b).*

Proof. By contradiction. Suppose that f changes sign only $r < k$ times, at the points $t_1 < t_2 < \cdots < t_r$. Then f will not change sign on each of the subintervals:

$$(a, t_1), (t_1, t_2), \cdots, (t_{r-1}, t_r), (t_r, b).$$

Let $p(x) = \prod_{i=1}^{r}(x - t_i) \in \mathbb{P}_r \subseteq \mathbb{P}_{k-1}$. Then $p(x)$ has the same sign properties as $f(x)$, and $f(x)p(x)$ does not change sign on the interval. Since $w > 0$ we get

$$\int_a^b w(x)f(x)p(x) \neq 0$$

which contradicts the assumption of the theorem. □

Corollary 5. *The orthogonal polynomial ϕ_k has exactly k distinct zeros in (a, b).*