NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 1

1 Assume that a function f is differentiable at a point x.

a) Prove that the derivative can be calculated as a *symmetric/central limit*:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}.$$

The discrete analogue of the symmetric limit is the central difference formula.

b) Does the existence of the symmetric limit at x imply that f is differentiable there?

2 Assume that f is twice differentiable at x. Show that the second derivative can be calculated as the following symmetric limit:

$$f''(x) = \lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}.$$

Hint: Use l'Hôpital's rule and Exercise 1.

3 Let $h > 0$. Then, if

i) $f \in C^1([x, x+h])$ and f'' exists and is bounded in $(x, x+h)$, we know that

$$\frac{f(x+h)-f(x)}{h} - f'(x) = \mathcal{O}(h) \qquad \text{as} \qquad h \to 0^+;$$

ii) $f \in C^1([x-h, x])$ and f'' exists and is bounded in $(x-h, x)$, we know that

$$\frac{f(x)-f(x-h)}{h} - f'(x) = \mathcal{O}(h) \qquad \text{as} \qquad h \to 0^+;$$

iii) $f \in C^2([x-h, x+h])$ and $f^{(3)}$ exists and is bounded in $(x-h, x+h)$, we know that

$$\frac{f(x+h)-f(x-h)}{2h} - f'(x) = \mathcal{O}(h^2) \qquad \text{as} \qquad h \to 0^+;$$

The definition of the $\mathcal{O}$ (bigoh) notation is as follows:

$$f(x) = \mathcal{O}(g(x)) \quad \text{as} \quad x \to \xi$$

if there are constants $M > 0$ and $\delta > 0$ such that $|f(x)| \leq M|g(x)|$ whenever $|x-\xi| \leq \delta$. The interpretation of this is that the growth of f is bounded above (within a factor M) by that of g when we are sufficiently close to ξ.

a) The boundedness-assumptions in i)–iii) cannot be dropped if we want to use the $\mathcal{O}$ (bigoh) notation. Why?

b) Verify i)–iii) by numerical experiments. Use the following test examples:

1) $f(x) = 1/x$ in $x_0 = 1$.

2) $f(x) = \sin x - \cos x$ in $x_0 = 1$ and $x_0 = \pi/4$.

3) $f(x) = \begin{cases} x^2 - 2x & \text{for } x > 1; \\ -x^2 + 2x - 2 & \text{for } x \leq 1 \end{cases}$ in $x_0 = 1$.

In each case, use stepsizes $h = 0.1 \times 2^{-i}$, where $i = 0, 1, \ldots, 10$. Present the results both as a table and as a convergence plot. Comment on the results: are they as expected? If not, explain why.

4 Let

$$y = \varphi_1(x) = \frac{1}{1+2x} - \frac{1-x}{1+x} \qquad \text{and} \qquad y = \varphi_2(x) = \frac{2x^2}{(1+2x)(1+x)}$$

be given.

a) Show that the two expressions are the same.

b) Set $x = 10^{-8}$ and compute y by the two formulas (in MATLAB). Do you get the same result? If not, which of the results will you trust, and why?

5 Find the condition numbers for the following functions:

a) x^α with α constant; b) $\ln x$; c) $x^{-1}\mathrm{e}^x$.

For which x are the problems ill-conditioned?

6 Define

$$x_n = \int_0^1 t^n (t+5)^{-1} \, \mathrm{d}t.$$

It is given that $x_{20} = 7.99752302823216383145210172138038148126351392 08\ldots \times 10^{-3}$.

a) Show that $x_0 = \ln 1.2$, and that $x_n = n^{-1} - 5x_{n-1}$ for $n \geq 1$. Compute x_n for $n = 1, 2, \ldots, 20$ using this reccurence formula. Is x_{20} correct?

b) Now use the reccurence formula backwards to find x_n for $n = 19, 18, \ldots, 0$. Is x_0 correct?

In both cases, explain the behaviour of the reccurence.