NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 2

1 First read the note on norms and inner products on $\mathbb{R}^m$. Then prove that

a) the ℓ^2-norm $\|x\|_2 = \left(\sum_{i=1}^m x_i^2\right)^{1/2}$ actually defines a norm on $\mathbb{R}^m$,

b) the equivalence $\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\|x\|_\infty$ holds for all $x \in \mathbb{R}^m$, and

c) the function $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\langle x, y \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 2 x_2 y_2$$

is an inner product on $\mathbb{R}^2$.

2 Suppose an explicit Runge–Kutta method has the Butcher-tableau

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

a) Write down the corresponding algorithm (Eq. (7), p. 11 in the note).

b) What is the increment function $\Phi(t_n, y_n; h)$ for this method? Show that Φ is Lipschitz in y, that is, establish the existence of a constant $M > 0$ such that

$$\|\Phi(t, y; h) - \Phi(t, \tilde{y}; h)\| \le M \|y - \tilde{y}\|$$

whenever f is Lipschitz in y and $h \le h_{\max}$ for some maximal stepsize $h_{\max}$. Find an expresssion for M (it will depend on the Lipschitz constant L of f and on $h_{\max}$).
Hint: Use the properties of a norm.

3 In the enclosed MATLAB-code `erk.m`, the three methods Euler, improved Euler and RK4 are implemented. In this exercise, you will use this to verify numerically that the methods are of order 1, 2 and 4, respectively. As test problems, use the scalar problem

$$y' = -2ty, \qquad y(0) = 1 \qquad \text{on} \quad t_0 = 0 \le t \le 1 = t_{\text{end}}.$$

with exact solution $y(t) = \mathrm{e}^{-t^2}$, and the system

$$y_1' = -y_2, \qquad y_2' = y_1, \qquad y_1(0) = 1,\ y_2(0) = 0, \qquad t_0 = 0 \le t \le 2\pi = t_{\text{end}},$$

with exact solution $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

a) Use stepsizes $h = 1/N$ with $N = 10, 20, 40, 80, \ldots$ and measure the error at the end of the integration using some appropriate norm (e.g. the 2-norm). Present the results as a convergence plot. Use reference lines to indicate the slopes of the curves and thereby the order of the method.

b) Include the method of Exercise 2 in your code, and determine its order experimentally.

4 The Duffing equation is given by

$$u'' + ku' - u(1-u^2) = A\cos(\omega t)$$

Set $k = 0.25$, $A = 0.4$, $\omega = 1$ and initial values $u(0) = u'(0) = 0$, and integrate up to $t_{\text{end}} = 100$.

a) Find and plot a reference solution by solving the problem by MATLABs solver ODE45, using very thight error tolerances:

```
    options = odeset('AbsTol', 1.e-12, 'RelTol', 1.e-12);
    [t, y] = ode45(@duffing, [0, 100], [0; 0], options);
    plot(t, y)
```

The problem has to be rewritten as a system of first-order equations first.

b) It may also be interesting to compare the solutions in the phase plane, that is, plot $u(t)$ versus $u'(t)$.

See for instance `http://www.scholarpedia.org/article/Duffing_oscillator` for more information about this equation.