NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 2

1 First read the note on norms and inner products on $\mathbb{R}^m$. Then prove that

a) the ℓ^2-norm $\|x\|_2 = \left(\sum_{i=1}^m x_i^2\right)^{1/2}$ actually defines a norm on $\mathbb{R}^m$,

Solution. We only consider the triangle inequality; the other properties are straightforward. Notice first that

$$\|x+y\|_2^2 = \sum_{i=1}^m (x_i+y_i)^2 = \|x\|_2^2 + \|y\|_2^2 + 2\sum_{i=1}^m x_i y_i.$$

We then use the elementary Young's inequality $2ab \le a^2 + b^2$ for $a, b \in \mathbb{R}$ (get it by expanding $0 \le (a-b)^2$) and a scaling trick to deal with the last term:

$$\begin{aligned}
2\sum_{i=1}^m x_i y_i &= \|x\|_2\|y\|_2 \cdot 2\sum_{i=1}^m \frac{x_i}{\|x\|_2}\frac{y_i}{\|y\|_2} \\
&\le \|x\|_2\|y\|_2 \cdot \sum_{i=1}^m \left[\left(\frac{x_i}{\|x\|_2}\right)^2 + \left(\frac{y_i}{\|y\|_2}\right)^2\right] \\
&= \|x\|_2\|y\|_2 \cdot \left[\frac{\|x\|_2^2}{\|x\|_2^2} + \frac{\|y\|_2^2}{\|y\|_2^2}\right] \\
&= 2\|x\|_2\|y\|_2.
\end{aligned}$$

This is the Cauchy-Schwarz inequality for the $\|\cdot\|_2$ norm. Hence,

$$\|x+y\|_2^2 \le \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2 = \left(\|x\|_2 + \|y\|_2\right)^2,$$

and so $\|x+y\|_2 \le \|x\|_2 + \|y\|_2$.

Another option is to show that $\langle x, y\rangle_2 = \sum_{i=1}^m x_i y_i$ is an inner product on $\mathbb{R}^m$ and appeal to the fact that inner products induce a natural norm $\|x\|_2 = \sqrt{\langle x, x\rangle_2}$.

b) the equivalence $\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\|x\|_\infty$ holds for all $x \in \mathbb{R}^m$, and

Solution. Let j be the index of the term of x with maximum absolute value. Then

$$\|x\|_\infty^2 = |x_j|^2 \le |x_j|^2 + \sum_{\substack{i=1 \\ i\ne j}}^m |x_i|^2 = \sum_{i=1}^m |x_i|^2 = \|x\|_2^2,$$

which establishes the first inequality. For the second case, we have $|x_i| \le \|x\|_\infty$ for all $i = 1, \ldots, m$, and so $\|x\|_2^2 = \sum_{i=1}^m |x_i|^2 \le m\|x\|_\infty^2$.

c) the function $\langle\cdot,\cdot\rangle\colon \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined by

$$\langle x,y\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$$

is an inner product on $\mathbb{R}^2$.

Solution. Let $x,y,z\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$. Then we have *symmetry*:

$$\langle x,y\rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle y,x\rangle,$$

linearity in the first argument:

$$\begin{aligned}\langle \alpha x+y,z\rangle &= (\alpha x_1+y_1)z_1 - (\alpha x_1+y_1)z_2 - (\alpha x_2+y_2)z_1 + 2(\alpha x_2+y_2)z_2\\ &= \alpha(x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2) + (y_1z_1 - y_1z_2 - y_2z_1 + 2y_2z_2)\\ &= \alpha\langle x,z\rangle + \langle y,z\rangle,\end{aligned}$$

and *positive definiteness*:

$$\langle x,x\rangle = x_1^2 - 2x_1x_2 + 2x_2^2 = (x_1-x_2)^2 + x_2^2 \ge 0,$$

where equality occurs if and only if $x_2=0$ and $x_1-x_2=0$, that is, also $x_1=0$.

$\boxed{2}$ Suppose an explicit Runge–Kutta method has the Butcher-tableau

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

a) Write down the corresponding algorithm (Eq. (7), p. 11 in the note).

Solution. The RK-method for the IVP $y'(t)=f(t,y(t))$ with datum $y(t_0)=y_0$ is

$$k_1 = f(t_n,y_n), \qquad k_3 = f(t_n+\tfrac{2h}{3}, y_n+\tfrac{2h}{3}k_2),$$

$$k_2 = f(t_n+\tfrac{h}{3}, y_n+\tfrac{h}{3}k_1), \qquad y_{n+1} = y_n + \tfrac{h}{4}(k_1+3k_3),$$

where $t_n = t_{n-1}+h$ and $h>0$ is the step size.

b) What is the increment function $\Phi(t_n,y_n;h)$ for this method? Show that Φ is Lipschitz in y, that is, establish the existence of a constant $M>0$ such that

$$\|\Phi(t,y;h)-\Phi(t,\tilde{y};h)\| \le M\|y-\tilde{y}\|$$

whenever f is Lipschitz in y and $h\le h_{\max}$ for some maximal stepsize $h_{\max}$. Find an expresssion for M (it will depend on the Lipschitz constant L of f and on $h_{\max}$).
Hint: Use the properties of a norm.

Solution. Backwards substitution from a) yields

$$\Phi(t_n, y_n; h) = \tfrac{1}{4}(k_1 + 3k_3) = \tfrac{1}{4}\Big[f(t_n, y_n) + 3f\Big(t_n + \tfrac{2h}{3}, y_n + \tfrac{2h}{3} f\big(t_n + \tfrac{h}{3}, y_n + \tfrac{h}{3} f(t_n, y_n)\big)\Big)\Big].$$

As regards the Lipschitz continuity of Φ in y, we use that $\|f(t,a) - f(t,b)\| \le L\|a-b\|$ for all t and a, b in some domain, and estimate with help of the triangle inequality:

$$\begin{aligned}
\|\Phi(t,y;h) - \Phi(t,\tilde{y};h)\| \le \tfrac{1}{4}\Big\{ & L\|y-\tilde{y}\| \\
& + 3L\left\| y + \tfrac{2h}{3} f\big(t + \tfrac{h}{3}, y + \tfrac{h}{3} f(t,y)\big) - \tilde{y} - \tfrac{2h}{3} f\big(t + \tfrac{h}{3}, \tilde{y} + \tfrac{h}{3} f(t,\tilde{y})\big)\right\| \Big\} \\
\le \tfrac{1}{4}\Big\{ & L\|y-\tilde{y}\| + 3L\Big[\|y-\tilde{y}\| + \tfrac{2h}{3} L \left\| y + \tfrac{h}{3} f(t,y) - \tilde{y} - \tfrac{h}{3} f(t,\tilde{y})\right\|\Big]\Big\} \\
\le \tfrac{1}{4}\Big\{ & L\|y-\tilde{y}\| + 3L\Big[\|y-\tilde{y}\| + \tfrac{2h}{3} L\big(\|y-\tilde{y}\| + \tfrac{h}{3} L\|y-\tilde{y}\|\big)\Big]\Big\} \\
= L & \Big[1 + \tfrac{1}{2} hL + \tfrac{1}{6}(hL)^2\Big] \|y-\tilde{y}\| \\
\le & \underbrace{L\Big[1 + \tfrac{1}{2} h_{\text{max}} L + \tfrac{1}{6}(h_{\text{max}} L)^2\Big]}_{:= M} \|y-\tilde{y}\|.
\end{aligned}$$

$\boxed{3}$ In the enclosed MATLAB-code `erk.m`, the three methods Euler, improved Euler and RK4 are implemented. In this exercise, you will use this to verify numerically that the methods are of order 1, 2 and 4, respectively. As test problems, use the scalar problem

$$y' = -2ty, \qquad y(0) = 1 \qquad \text{on} \quad t_0 = 0 \le t \le 1 = t_{\text{end}}.$$

with exact solution $y(t) = \mathrm{e}^{-t^2}$, and the system

$$y_1' = -y_2, \qquad y_2' = y_1, \qquad y_1(0) = 1,\ y_2(0) = 0, \qquad t_0 = 0 \le t \le 2\pi = t_{\text{end}},$$

with exact solution $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

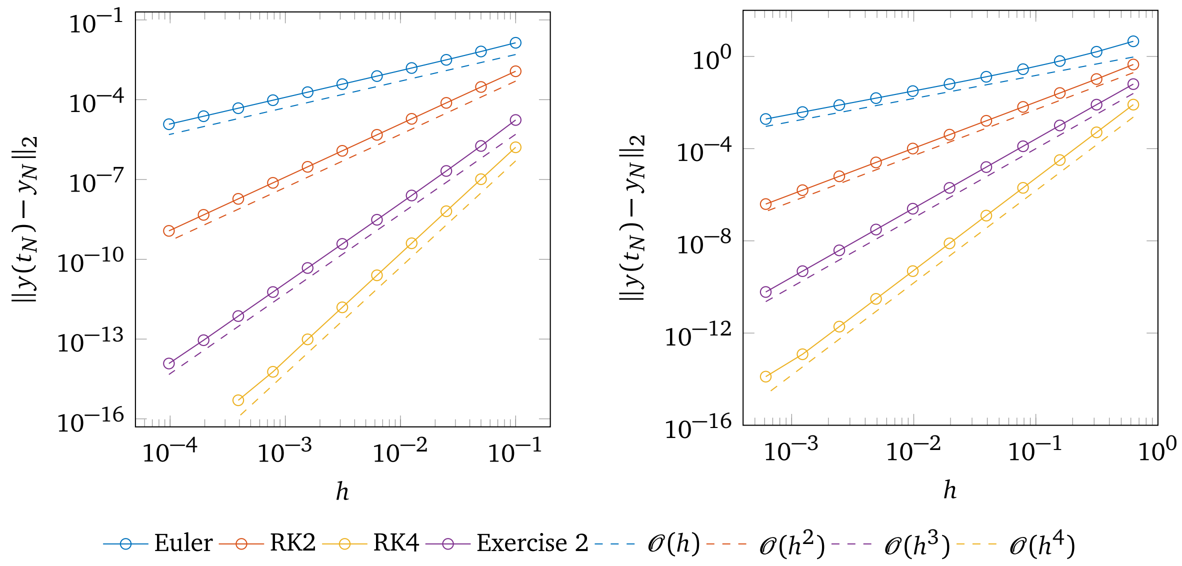
a) Use stepsizes $h = 1/N$ with $N = 10, 20, 40, 80, \dots$ and measure the error at the end of the integration using some appropriate norm (e.g. the 2-norm). Present the results as a convergence plot. Use reference lines to indicate the slopes of the curves and thereby the order of the method.

b) Include the method of Exercise 2 in your code, and determine its order experimentally.

Solution. In Figure 1 we have plotted the numerical global errors for the 4 one-step methods using the 2-norm. As is seen, the orders seem to be 1 for Euler, 2 for RK2, 4 for RK4 and, finally, 3 for the method in Exercise 2. The implementation of the latter in `erk.m` can be as follows:

```
elseif strcmpi(met,'exercise2')
    for n=1:N
        yn = y(:,n);
```

```
        tn = t(n);
        k1 = f(tn,yn);
        k2 = f(tn+(h/3),yn+(h/3)*k1);
        k3 = f(tn+(2*h/3),yn+(2*h/3)*k2);
        y(:,n+1) = yn + h*(k1+3*k3)/4;
    end
end
```



(a) Scalar case.

(b) System case.

Figure 1: Order of convergence-plots for the 4 one-step methods.

$\boxed{4}$ The Duffing equation is given by

$$u'' + ku' - u(1-u^2) = A\cos(\omega t)$$

Set $k = 0.25$, $A = 0.4$, $\omega = 1$ and initial values $u(0) = u'(0) = 0$, and integrate up to $t_{\text{end}} = 100$.

a) Find and plot a reference solution by solving the problem by MATLABs solver ODE45, using very thight error tolerances:

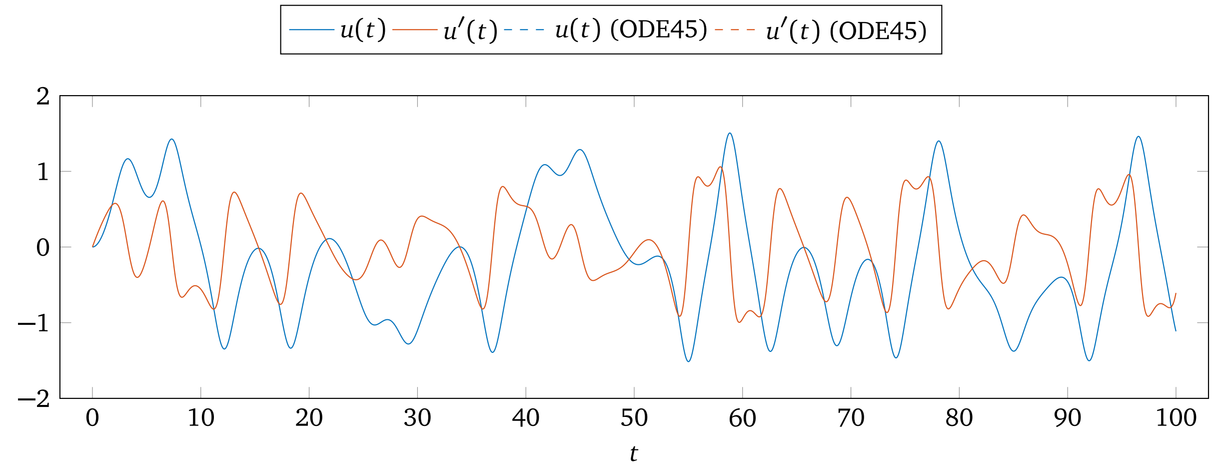
```
    options = odeset('AbsTol', 1.e-12, 'RelTol', 1.e-12);
    [t, y] = ode45(@duffing, [0, 100], [0; 0], options);
    plot(t, y)
```

The problem has to be rewritten as a system of first-order equations first.

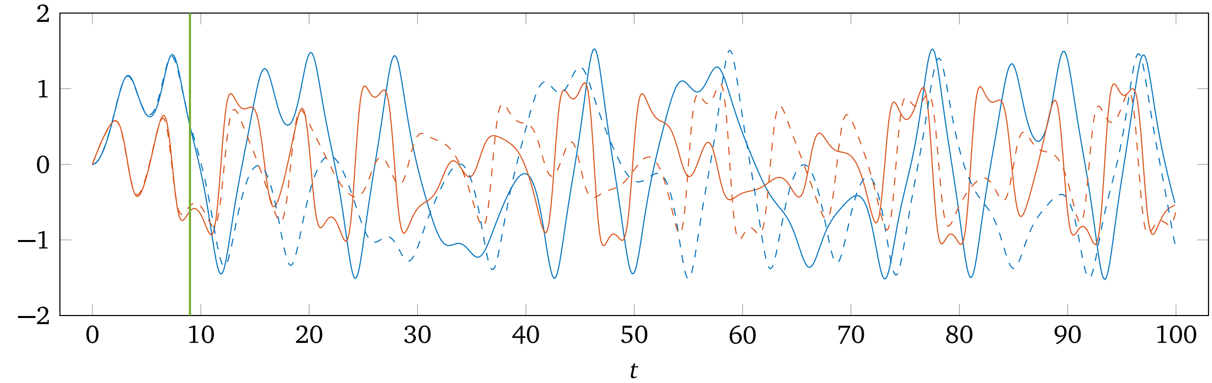
Solution. We rewrite the Duffing equation into

$$\begin{cases} u' = v \\ v' = A\cos(\omega t) - kv + u(1-u^2), \end{cases}$$

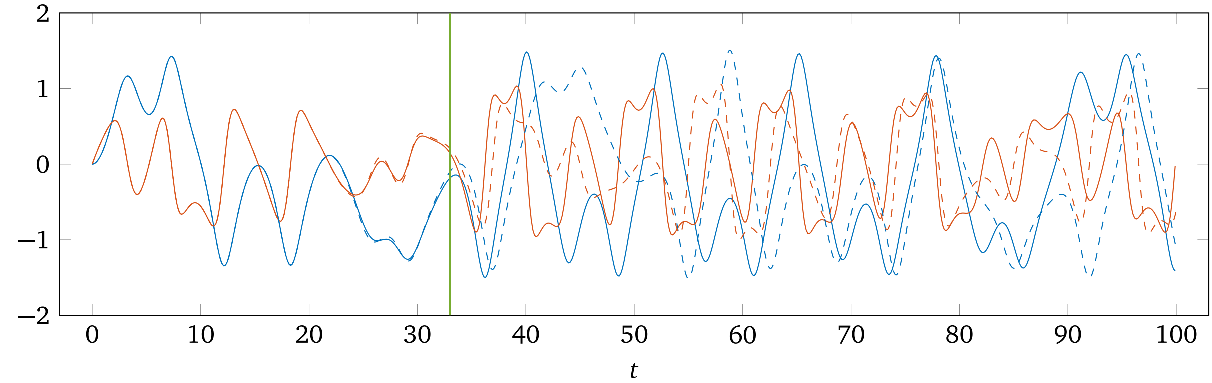
with $u(0) = 0 = v(0)$, and in Figure 2a the numerical solution from MATLABs ODE45 is shown.



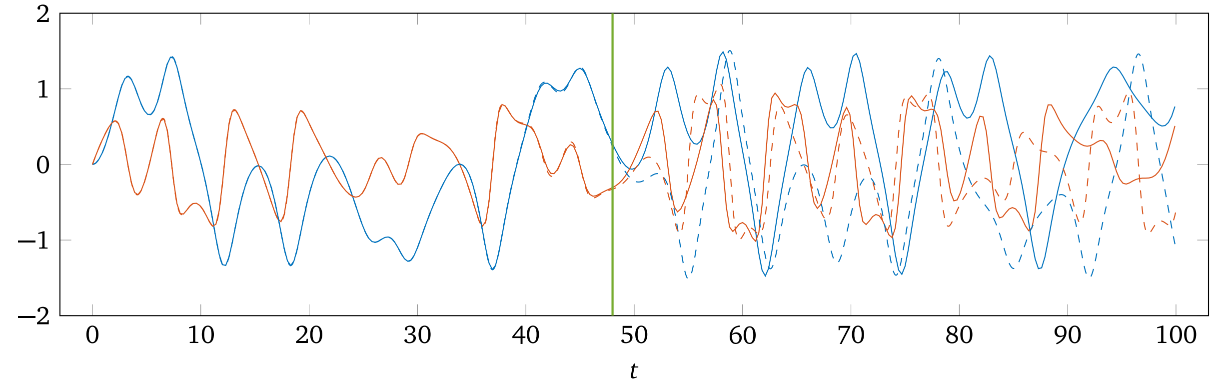
(a) ODE45.



(b) Euler.



(c) RK2 (improved Euler).



(d) RK4.

Figure 2: Numerical solutions of the Duffing equation using various schemes. The breakdowns of Euler, RK2 and RK4 are indicated roughly by the green lines.

b) Now, solve the same problem using Euler's method with $N = 4000$ steps, RK2 with $N = 2000$ steps and RK4 with $N = 1000$ steps. In this case, they will all use 4000 f-evaluations for the integration. Plot the solutions and compare with the reference solution computed above. What do you observe? Does it pay off to use a higher-order method like RK4 in this case?

Solution. The numerical solutions are found in Figure 2b–d. In addition, we have added green lines indicating roughly where they start to break down. As we can observe, increasing the order results in good approximations for longer time periods.

c) It may also be interesting to compare the solutions in the phase plane, that is, plot $u(t)$ versus $u'(t)$.

Solution. Figure 3 displays the phase-plane view of the numerical solutions. We notice that, although Euler, RK2 and RK4 do not get the complete picture, their solutions at least stay (reasonably) inside the exact outer region of Figure 3a.

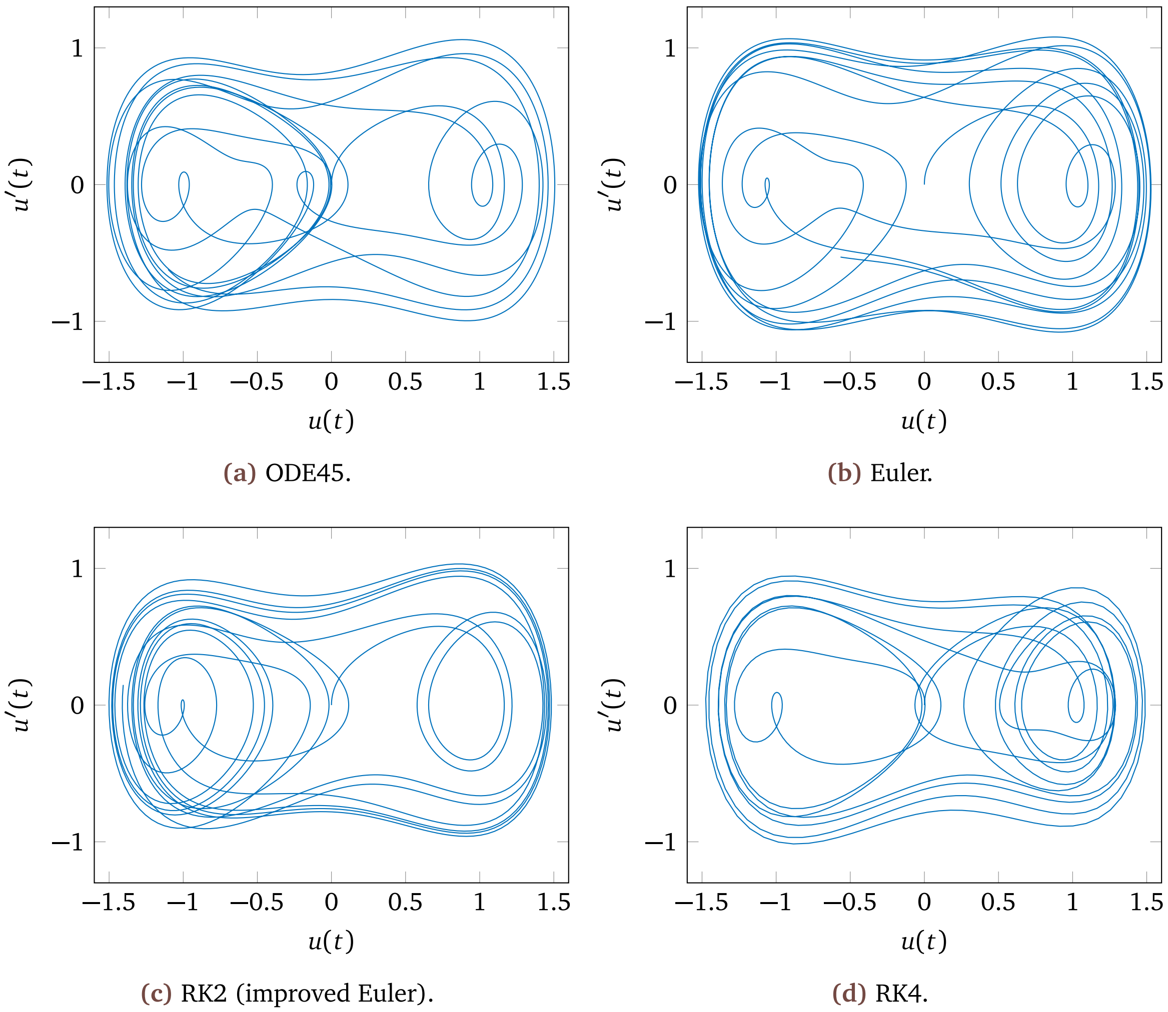


Figure 3: Phase plane-view of numerical solutions of the Duffing equation using various schemes.

See for instance `http://www.scholarpedia.org/article/Duffing_oscillator` for more information about this equation.