NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 3

1 Kutta's method from 1901 is definitely the most famous of all explicit Runge–Kutta pairs. Its Butcher tableau is given by

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

a) Verify that the method is of order 4 by confirming that it obeys all the eight order conditions.

b) A captivating thought would be to find another set of weights $\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3, \widehat{b}_4\}$, such that the associated method has order 3. This we can use for error estimates and stepsize control. Try to find such a set of weights.

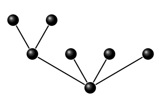
2 a) Show that an explicit Runge–Kutta method of order 3 with 3 stages has to satisfy

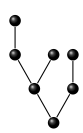
$$3\,a_{32}\,c_2^2 - 2\,a_{32}\,c_2 - c_2\,c_3 + c_3^2 = 0.$$

b) Characterize all 3rd-order explicit Runge–Kutta methods with 3 stages which satisfy $a_{31} = 0$, that is, which satisfy $a_{32} = c_3$. How many free parameters are there?

c) Find all explicit methods of order 2 which have the same A matrix as the method(s) in b), but with weights $\{\widehat{b}_1, \widehat{b}_2, \widehat{b}_3\}$ that also satisfy $\widehat{b}_3 = 0$.

3 Write down the order conditions corresponding to the following rooted trees:





Also, write down the order of the trees.

4 Draw all rooted trees of order 5 (there are 9 different ones).

$\boxed{5}$ a) Implement an adaptive Runge–Kutta solver based on the Bogacki-Shampine pair:

$$\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
\hline
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}$$

Use $P = 0.9$ as a pessimist factor and measure the local error estimate in the 2-norm.

b) Test your program on the Lotka-Volterra equation

$$\begin{aligned} y_1' &= y_1 - y_1 y_2 \\ y_2' &= y_1 y_2 - 2y_2, \end{aligned}$$

on the interval $[0, 20]$ with initial conditions $y_1(0) = 1$ and $y_2(0) = 2$. Let $h_0 = 1/20$ be the starting stepsize and choose the tolerance 1×10^{-5}. Make a plot of the solution as well as of the stepsizes h_n. Do h_n behave as expected?

c) Plot the ratio $\|e_N\|_2/\mathtt{TOL}$ between the tolerance and the global error, where e_N is the global error at the end of the interval. For example, you can choose a series of tolerances 2^{-i}, where $i = 1, \dots, 20$. In order to compute the global error, compare with an “exact” reference solution using MATLABs ODE45 integrator. What do you observe?