TMA4215 Numerical Mathematics — Fall 2016

PROBLEM SET 4

NTNU

1 The Butcher tableau for the Bogacki–Shampine pair is

$$
\begin{array}{c|cccc}
0 & & & & \\
1/2 & 1/2 & & & \\
3/4 & 0 & 3/4 & & \\
1 & 2/9 & 1/3 & 4/9 & \\
\hline
 & 2/9 & 1/3 & 4/9 & 0 \\
\hline
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}
$$

Find the stability functions of the two methods, and plot the stability regions with MATLAB, writing something similar as:

```
% Plot the stabilty domain for a given stability function.
% Domain: [-a, a, -b, b]
a = 4; b = 4;
[x, y] = meshgrid(linspace(-a, a), linspace(-b, b));
z = x + i*y;

% Stability function.
R = abs(1 + z + z.^2/2);

% Make the plot.
contourf(x, y, R, [1 1], 'k')
axis equal, axis([-a a -b b]), grid on
hold on
plot([-a, a], [0, 0], 'k', 'LineWidth', 1);
plot([0, 0], [-a, a], 'k', 'LineWidth', 1);
```

2 a) Find the eigenvalues of the matrix

$$M = \begin{bmatrix} -10 & -10 \\ 40 & -10 \end{bmatrix}.$$

b) Assume that you are to solve the differential equation

$$y' = My, \qquad y(0) = y_0$$

using the improved Euler method. What is the largest stepsize $h_{\max}$ you can use?

c) Solve the equation

$$y' = My + g, \qquad 0 \le t \le 10$$

with

$$g(t) = \left[\sin t, \cos t\right]^\top \qquad \text{and} \qquad y(0) = \left[\frac{5210}{249401}, \frac{20259}{249401}\right]^\top$$

with help of the `erk.m`-function from Problem Set 2. Choose stepsizes a little smaller than and a little larger than $h_{\max}$. What do you observe?

d) Solve the equation in c) by the backward Euler method. Use different stepsizes, *e.g.* 0.1 and 0.5. Do you observe any stepsize restrictions due to stability in this case?

$\boxed{3}$ A method is *A*-stable if $|R(z)| \le 1$ for all $z \in \mathbb{C}^-$. The stability function R is always a rational function, that is, $R(z) = P(z)/Q(z)$ where P and Q are polynomials in z. By using the maximum modulus principle it is possible to show that a method is *A*-stable if and only if

- ⋆ $R(z)$ does not have poles in $\mathbb{C}^-$ (poles are zeros of $Q(z)$),
- ⋆ $|R(yi)|^2 \le 1$ for all $y \in \mathbb{R}$.

a) Use this to show that the 3rd-order implicit Runge–Kutta method

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

is *A*-stable. Plot the stability region.

b) Find the system of nonlinear equations that must be solved when taking a step with this method applied to the van der Pol equation

$$y'' - \mu(1 - y^2)y' + y = 0,$$

where $\mu > 0$ is a known constant.

Help: Write the equation as a first-order system.

(Later, we will discuss how to solve such systems of nonlinear equations.)

$\boxed{4}$ Which of the following linear multistep methods is/are convergent? State the order p and the error constant C_{p+1} for each method.

a) $y_{m+2} + y_{m+1} - 2y_m = \frac{h}{4}\left[f(x_{m+2}, y_{m+2}) + 8f(x_{m+1}, y_{m+1}) + 3f(x_m, y_m)\right]$.

b) $y_{m+3} + \frac{1}{4}y_{m+2} - \frac{1}{2}y_{m+1} - \frac{3}{4}y_m = \frac{h}{8}\left[19f(x_{m+2}, y_{m+2}) + 5f(x_m, y_m)\right]$.

c) $y_{m+2} - y_{m+1} = \frac{h}{3}\left[3f(x_{m+1}, y_{m+1}) - 2f(x_m, y_m)\right]$.