NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 4

1 The Butcher tableau for the Bogacki–Shampine pair is

0				
1/2	1/2			
3/4	0	3/4		
1	2/9	1/3	4/9	
	2/9	1/3	4/9	0
	7/24	1/4	1/3	1/8

Find the stability functions of the two methods, and plot the stability regions with MATLAB, writing something similar as:

```
% Plot the stabilty domain for a given stability function.
% Domain: [-a, a, -b, b]
a = 4; b = 4;
[x, y] = meshgrid(linspace(-a, a), linspace(-b, b));
z = x + i*y;

% Stability function.
R = abs(1 + z + z.^2/2);

% Make the plot.
contourf(x, y, R, [1 1], 'k')
axis equal, axis([-a a -b b]), grid on
hold on
plot([-a, a], [0, 0], 'k', 'LineWidth', 1);
plot([0, 0], [-a, a], 'k', 'LineWidth', 1);
```

Solution. The stability function $R(z)$ of a method is defined from $y_{n+1} = R(z)\,y_n$, where $z = h\lambda$, when we apply the method to the linear test equation $y' = \lambda y$, with $\lambda \in \mathbb{C}$. For the first method in the pair we have

$$k_1 = \lambda y_n, \quad k_2 = \lambda\left(y_n + \tfrac{1}{2}hk_1\right), \quad k_3 = \lambda\left(y_n + \tfrac{3}{4}hk_2\right) \quad \text{and} \quad y_{n+1} = y_n + h\left(\tfrac{2}{9}k_1 + \tfrac{1}{3}k_2 + \tfrac{4}{9}k_3\right),$$

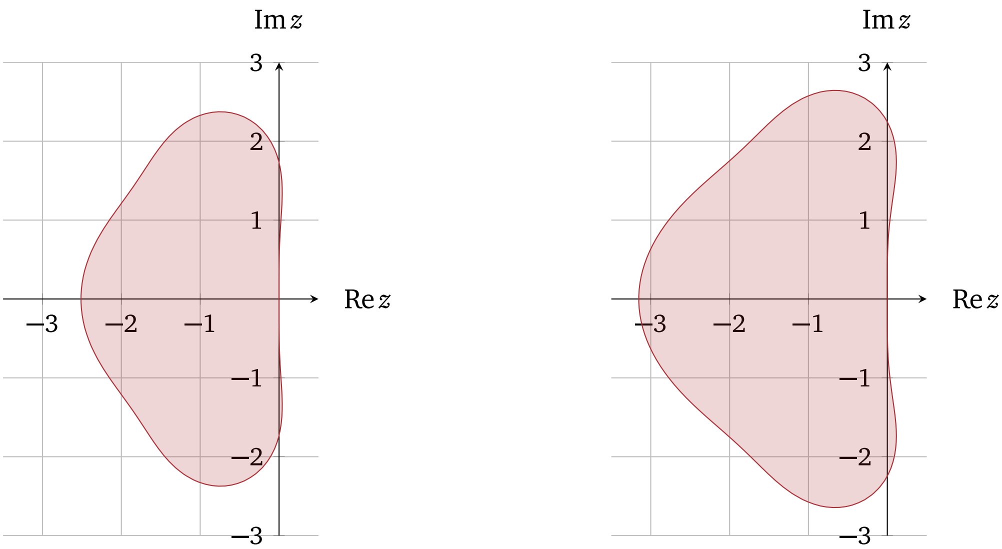
which gives the stability function

$$R_1(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3.$$

We could have deduced this immediately since the 3-stage explicit RK-method is of 3rd order. Similarly, the second method yields

$$R_2(z) = 1 + z + \tfrac{1}{2}z^2 + \tfrac{3}{16}z^3 + \tfrac{1}{48}z^4,$$

and in Figure 1 we have plotted their absolute stability regions, that is, all $z \in \mathbb{C}$ for which their modulus is less than or equal to 1.



(a) $|R_1(z)| \leq 1$. **(b)** $|R_2(z)| \leq 1$.

Figure 1: Absolute stability regions (colored).

$\boxed{2}$ a) Find the eigenvalues of the matrix

$$M = \begin{bmatrix} -10 & -10 \\ 40 & -10 \end{bmatrix}.$$

Solution. The eigenvalues are $\lambda_{1,2} = -10 \pm 20\mathrm{i}$.

b) Assume that you are to solve the differential equation

$$y' = My, \qquad y(0) = y_0$$

using the improved Euler method. What is the largest stepsize $h_{\max}$ you can use?

Solution. The stability function is given by

$$R(z) = 1 + z + \tfrac{1}{2}z^2.$$

We must have

$$|R(h\lambda)| \leq 1$$

for the numerical solution to be stable for all eigenvalues of M. In our case, this means that

$$\begin{aligned} \big|R\big((-10 \pm 20\mathrm{i})h\big)\big|^2 &= R\big((-10 \pm 20\mathrm{i})h\big) \cdot R\big((-10 \mp 20\mathrm{i})h\big) \\ &= 1 - 20h + 200h^2 - 5000h^3 + 62500h^4 \leq 1, \end{aligned}$$

which is satisfied when $0 \leq h \leq h_{\max} \lessapprox 0.086036$.

c) Solve the equation

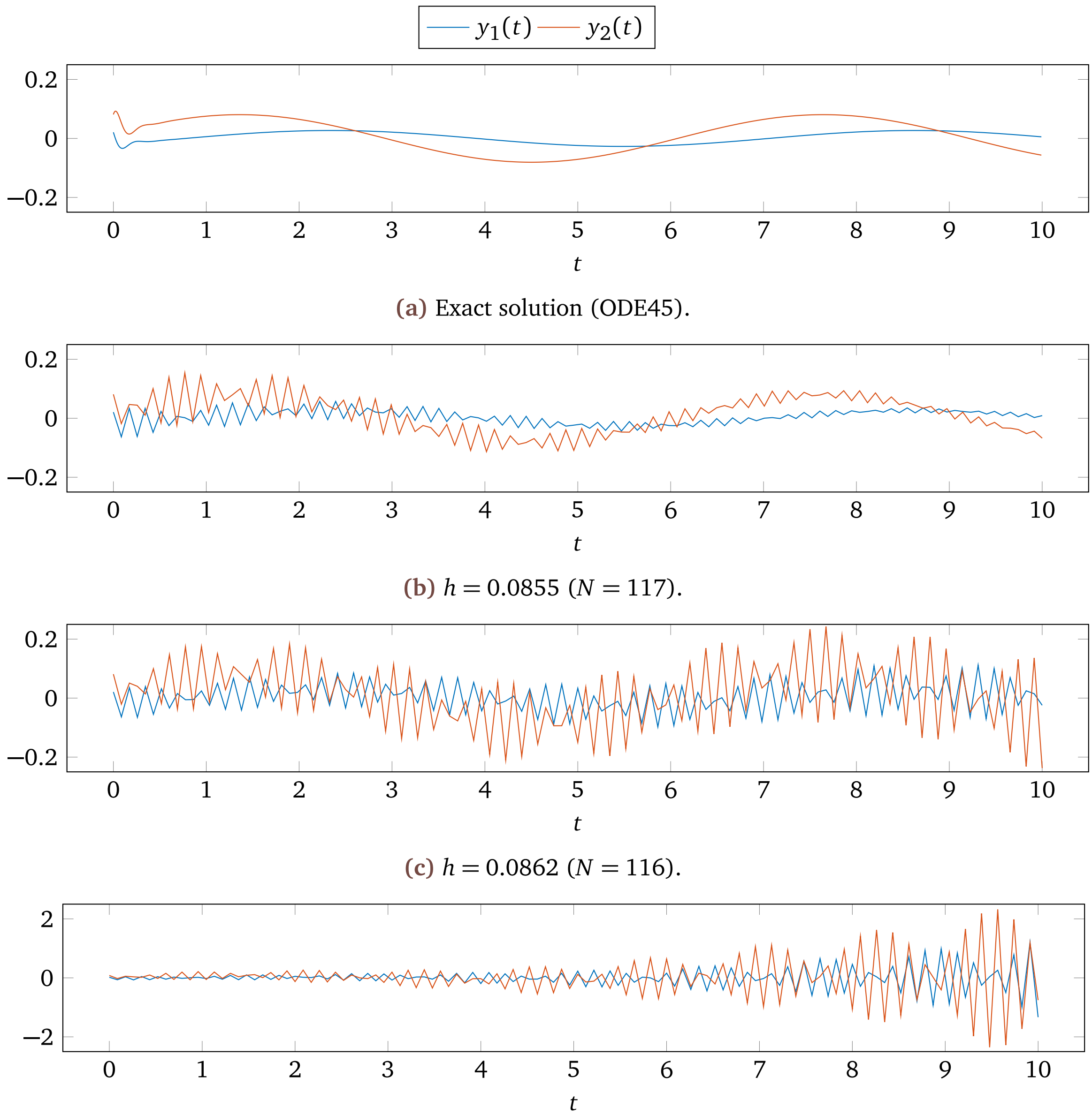
$$y' = My + g, \qquad 0 \le t \le 10$$

with

$$g(t) = \left[\sin t, \cos t\right]^\top \qquad \text{and} \qquad y(0) = \left[\frac{5210}{249401}, \frac{20259}{249401}\right]^\top$$

with help of the `erk.m`-function from Problem Set 2. Choose stepsizes a little smaller than and a little larger than $h_{\max}$. What do you observe?

Solution. In Figure 2 we have plotted the exact solution and three attempts using the improved Euler method with stepsizes close to $h_{\max}$. All cases show instability, and Figure 2d is a complete breakdown.



(a) Exact solution (ODE45).

(b) $h = 0.0855$ ($N = 117$).

(c) $h = 0.0862$ ($N = 116$).

(d) $h = 0.087$ ($N = 115$).

Figure 2: Exact solution, and disastrous numerical solutions using the improved Euler method with various stepsizes a little below and above the critical stepsize $h_{\max}$.

d) Solve the equation in c) by the backward Euler method. Use different stepsizes, *e.g.* 0.1 and 0.5. Do you observe any stepsize restrictions due to stability in this case?

Solution. A simple implementation of the backward Euler method can be found on the webpage. We do not observe any stepsize restrictions, which is in agreement with the backward Euler method being A-stable.

$\boxed{3}$ A method is A-stable if $|R(z)| \leq 1$ for all $z \in \mathbb{C}^-$. The stability function R is always a rational function, that is, $R(z) = P(z)/Q(z)$ where P and Q are polynomials in z. By using the maximum modulus principle it is possible to show that a method is A-stable if and only if

- $R(z)$ does not have poles in $\mathbb{C}^-$ (poles are zeros of $Q(z)$),
- $|R(y\mathrm{i})|^2 \leq 1$ for all $y \in \mathbb{R}$.

a) Use this to show that the 3rd-order implicit Runge–Kutta method

$$\begin{array}{c|cc} 1/3 & 5/12 & -1/12 \\ 1 & 3/4 & 1/4 \\ \hline & 3/4 & 1/4 \end{array}$$

is A-stable. Plot the stability region.

Solution. The RK-scheme applied to $y' = \lambda y$ is

$$k_1 = \lambda\left(y_n + \tfrac{1}{12}h(5k_1 - k_2)\right), \quad k_2 = \lambda\left(y_n + \tfrac{1}{4}h(3k_1 + k_2)\right) \quad \text{and} \quad y_{n+1} = y_n + \tfrac{1}{4}h(3k_1 + k_2).$$

This gives $y_{n+1} = k_2/\lambda$ and

$$3k_1 + k_2 = 4\lambda y_n + 2zk_1,$$

where $z = \lambda h$, so that

$$k_1 = \frac{\lambda y_n\left(1 - \frac{1}{3}z\right)}{1 - \frac{2}{3}z + \frac{1}{6}z^2}.$$

Putting things together we find that the stability function is

$$R(z) = \frac{k_2}{\lambda y_n} = \frac{1 + \frac{1}{3}z}{1 - \frac{2}{3}z + \frac{1}{6}z^2},$$

and the absolute stability region is shown in Figure 3.

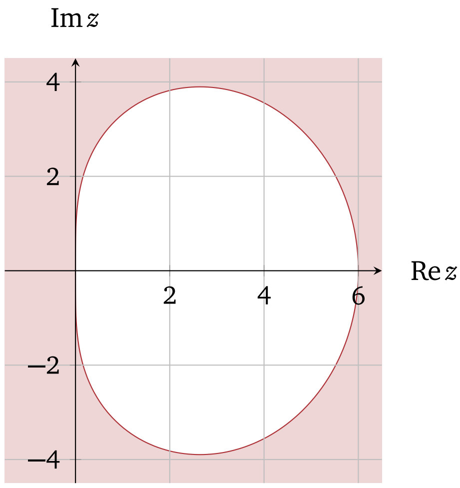


Figure 3: Stability region (colored).

A routine calculation yields that R has poles at $2 \pm \sqrt{2}\mathrm{i}$. Hence, condition 1 is satisfied. Additionally,

$$|R(y\mathrm{i})|^2 = R(y\mathrm{i})R(-y\mathrm{i}) = \frac{4y^2 + 36}{y^4 + 4y^2 + 36} \leq 1$$

for all $y \in \mathbb{R}$, so yes, the method is A-stable (this we also see from Figure 3).

b) Find the system of nonlinear equations that must be solved when taking a step with this method applied to the van der Pol equation

$$y'' - \mu(1-y^2)y' + y = 0,$$

where $\mu > 0$ is a known constant.

Help: Write the equation as a first-order system.

(Later, we will discuss how to solve such systems of nonlinear equations.)

Solution. We write van der Pol's equation as

$$\begin{cases} y' = v, \\ v' = \mu(1-y^2)v - y, \end{cases} \qquad \text{or, in vector notation,} \qquad \begin{bmatrix} y \\ v \end{bmatrix}' = f\left(t, \begin{bmatrix} y \\ v \end{bmatrix}\right) = \begin{bmatrix} v \\ \mu(1-y^2)v - y \end{bmatrix}.$$

Let $k_{y,1}$ and $k_{v,1}$ denote the components of the k_1 vector, and similarly for k_2. Then for each step the following system of equations must be solved:

$$k_{y,1} = V_1, \qquad k_{v,1} = \mu\left(1-Y_1^2\right)V_1 - Y_1,$$
$$k_{y,2} = V_2, \qquad k_{v,2} = \mu\left(1-Y_2^2\right)V_2 - Y_2,$$

with

$$Y_1 = y_n + \tfrac{1}{12}h\left(5k_{y,1} - k_{y,2}\right), \qquad Y_2 = y_n + \tfrac{1}{4}h\left(3k_{y,1} + k_{y,2}\right),$$
$$V_1 = v_n + \tfrac{1}{12}h\left(5k_{v,1} - k_{v,2}\right), \qquad V_2 = v_n + \tfrac{1}{4}h\left(3k_{v,1} + k_{v,2}\right).$$

The solution is updated by

$$y_{n+1} = Y_2, \qquad v_{n+1} = V_2.$$

$\boxed{4}$ Which of the following linear multistep methods is/are convergent? State the order p and the error constant C_{p+1} for each method.

Solution. By Dahlquist's equivalence theorem, an LMM

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = \sum_{l=0}^{k} \beta_l f(x_{m+l}, y_{m+l})$$

is convergent if and only if it is consistent and zero-stable. Let

$$C_0 = \sum_{l=0}^{k} \alpha_l \qquad \text{and} \qquad C_q = \frac{1}{q!}\sum_{l=0}^{k}\left(l^q \alpha_l - q l^{q-1}\beta_l\right) \quad \text{for} \quad q = 1, 2, \ldots.$$

Then the method is consistent if $C_0 = C_1 = 0$ and of order p if

$$C_0 = C_1 = \cdots = C_p = 0 \qquad \text{but} \qquad C_{p+1} \neq 0.$$

In addition, an LMM is zero-stable if and only if all the roots of the characteristic polynomial $\rho(r) = \sum_{l=0}^{k} \alpha_l r^l$ lie in the closed unit disk $\mathbb{D} = \{r \in \mathbb{C} : |r| \leq 1\}$ and only simple roots are on the boundary $\partial\mathbb{D} = \{r \in \mathbb{C} : |r| = 1\}$.

a) $y_{m+2} + y_{m+1} - 2y_m = \frac{h}{4}\left[f(x_{m+2}, y_{m+2}) + 8f(x_{m+1}, y_{m+1}) + 3f(x_m, y_m)\right]$.

Solution. Routine calculations yield that $C_0 = C_1 = C_2 = C_3 = 0$ and $C_4 = 1/24$. Hence, the method is consistent and of order 3. The roots of $\rho(r) = r^2 + r - 2$ are $r_1 = 1$ and $r_2 = -2$, so the LMM is not zero-stable, and therefore not convergent.

b) $y_{m+3} + \frac{1}{4}y_{m+2} - \frac{1}{2}y_{m+1} - \frac{3}{4}y_m = \frac{h}{8}\left[19f(x_{m+2}, y_{m+2}) + 5f(x_m, y_m)\right]$.

Solution. The method is of order 3, with $C_4 = 17/48$. Moreover, $\rho(r) = r^3 + \frac{1}{4}r^2 - \frac{1}{2}r - \frac{3}{4}$ has roots $r_1 = 1$ and $r_{2,3} = \left(-5 \pm \mathrm{i}\sqrt{23}\right)/8$. Hence, it is zero-stable and thus also convergent.

c) $y_{m+2} - y_{m+1} = \frac{h}{3}\left[3f(x_{m+1}, y_{m+1}) - 2f(x_m, y_m)\right]$.

Solution. Since $C_1 = 2/3$, the LMM is of order 0. Hence, consistency fails and it cannot be convergent. It is, however, zero-stable.