NTNU

TMA4215 Numerical Mathematics — Fall 2016

Problem Set 5

1 Let $\{t_i\}_{i=0}^n$ be a set of equidistant nodes, so that $t_i = t_0 + ih$ for $i = 1, 2, \ldots, n$, where h is the distance between the nodes. Given a function f defined on an interval containing the nodes, with $f_i := f(t_i)$ for $i = 0, 1, \ldots, n$, we can introduce *forward differences* recursively by

$$\Delta^0 f_i = f_i, \qquad \Delta f_i = f_{i+1} - f_i \qquad \text{and} \qquad \Delta^j f_i = \Delta\big(\Delta^{j-1} f_i\big) = \Delta^{j-1} f_{i+1} - \Delta^{j-1} f_i$$

for $i = 0, \ldots, n$ and $j = 0, 1, \ldots, n-i$. For example,

$$\Delta^2 f_0 = f_2 - 2f_1 + f_0 \qquad \text{and} \qquad \Delta^3 f_0 = f_3 - 3f_2 + 3f_1 - f_0.$$

Now let $t = t_0 + sh$, where $s \in \mathbb{R}$, be any point. In this exercise we want to establish that the polynomial p_n of degree n interpolating f in the nodes $\{t_i\}$ can be written as

$$p_n(t) = p_n(t_0 + sh) = f_0 + \sum_{j=1}^{n} \binom{s}{j} \Delta^j f_0, \tag{1}$$

where

$$\binom{s}{j} := \frac{s(s-1)\cdots(s-j+1)}{j!}$$

is a generalized binomial coefficient. Equation (1) is called *Newton's forward difference formula*.

a) Show by induction that

$$f[t_0, t_1, \ldots, t_j] = \frac{\Delta^j f_0}{j!\,h^j} \qquad \text{for} \qquad j = 1, \ldots, n, \tag{$\star$}$$

where $f[t_0, t_1, \ldots, t_j]$ is a *forward divided difference* defined from

$$f[t_j] = f_j \quad \text{for} \quad j \in \{0, \ldots, n\}$$

and

$$f[t_i, \ldots, t_{i+j}] = \frac{f[t_{i+1}, \ldots, t_{i+j}] - f[t_i, \ldots, f_{i+j-1}]}{t_{i+j} - t_i}$$

for $j \in \{1, \ldots, n\}$ and $i \in \{0, \ldots, n-j\}$.

b) Show that

$$\prod_{i=0}^{j-1} (t - t_i) = j!\,h^j \binom{s}{j} \qquad \text{for} \qquad j = 1, \ldots, n.$$

c) Use the results from a) and b) to prove (1).

Comment: Equivalently, it is possible to show *Newton's backward difference formula*. Backward differences on the sequence $\{f_i\}$ are defined by

$$\nabla^0 f_i = f_i, \qquad \nabla f_i = f_i - f_{i-1}, \qquad \nabla^j f_i = \nabla^{j-1} f_i - \nabla^{j-1} f_{i-1}, \qquad j = 1, 2, \ldots$$

Newton's backward difference formula is given by

$$p_n(t) = p_n(t_n + sh) = f_n + \sum_{j=1}^{n} (-1)^j \binom{-s}{j} \nabla^j f_n. \tag{2}$$

This expression is used to develop Adams methods; see the note on the webpage. In the next exercise, it is used to develop the so-called BDF-methods.

2 The *backward differentiation formulas* (BDFs) is a family of implicit linear k-step methods to the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0,$$

where we assume that the starting values $y_1, \ldots, y_{k-1}$ (in addition to y_0) are given with sufficient accuracy. To construct them, we first use (2) to find the polynomial $q \in \mathbb{P}_k$ interpolating all the previously computed points plus the new one, that is, the points

$$(t_{n-k+1}, y_{n-k+1}),\ \ldots,\ (t_n, y_n),\ (t_{n+1}, y_{n+1}).$$

Then we replace y by q in the ODE and evaluate the corresponding equation at t_{n+1}, which yields the BDF

$$q'(t_{n+1}) = f(t_{n+1}, y_{n+1}).$$

a) Establish that a k-step BDF can be written as

$$\sum_{j=1}^{k} \delta_j^\star \nabla^j y_{n+1} = h f_{n+1},$$

and find an expression for $\delta_j^\star$.

b) Write down the methods for $k = 1, 2$ and 3 and check if they are zero-stable. Moreover, find their orders and the corresponding error constants.

c) Based on b), guess what the error constant would be for a general k-step BDF, and try to prove it.

NB! BDF-methods with $k > 6$ are not zero-stable, and thus not convergent.