TMA4215 Numerical Mathematics — Fall 2016

Problem Set 6

1 Find an approximation to the integral

$$\int_1^{1.5} x^2 \ln x \, dx$$

by Romberg integration (see the note on the webpage). Do the first 3 rows by hand, starting with $h = 0.5$. Then rewrite the MATLAB code `extrapolation.m` for doing Romberg integration, and apply the code on the integral above. How many rows in the extrapolation table are needed before no better accuracy can be obtained?

2 a) Let $F(h)$ be our numerical approximation to some solution Q, and assume we have an error expansion given by

$$F(h) = Q + C_1 h + C_2 h^2 + C_3 h^3 + \cdots,$$

where the constants C_k are independent of h. Explain how you can make an extrapolation table based on the stepsize sequence $\{h/j\}$, $j = 1, 2, \ldots$.

Write a MATLAB code to test your algorithm on the forward difference approximation to the derivative of $f(x)$ in some point x_0. In this case

$$F(h) = \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{and} \qquad Q = f'(x_0).$$

b) It can be proved that, using the forward Euler method

$$y_{n+1} = y_n + h f(t_n, y_n) \qquad \text{with} \qquad n = 0, \ldots, N-1$$

to solve the IVP $y' = f(t, y)$ and $y(t_0) = y_0$ from t_0 to $t_0 + H$, the global error can be expressed as

$$y_N = y(t_0 + H) + C_1 h + C_2 h^2 + C_3 h^3 + \cdots, \qquad \text{where} \qquad h = \frac{H}{N}.$$

See if you can figure out how the extrapolation algorithm developed in a) applied to this problem can be expressed as explicit Runge–Kutta methods of arbitrary high order (with a lot of stages). The stepsize used in these RK-methods is H.

Hint: Do one extrapolation step at a time starting out with the explicit Euler method and find the ERK formulation of the 2nd-order approximation, then of the 3rd-order, *etc.* Identify the function evaluations, since

$$k_i = f\Big(t_0 + c_i H, y_0 + H \sum_{j=1}^{i-1} a_{ij} k_j\Big).$$

3 Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$.

a) Show that the spectral radius

$$\rho(A) = \max_{i=1}^{n} |\lambda_i|,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, is not a norm on $\mathbb{R}^{n\times n}$.

b) Prove that the *Frobenius norm*

$$\|A\|_{\mathrm{F}} = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} = \sqrt{\operatorname{tr}(AA^\top)}$$

is a norm on $\mathbb{R}^{n\times n}$.

c) Establish that $\|\cdot\|_{\mathrm{F}}$ is submultiplicative, that is,

$$\|AB\|_{\mathrm{F}} \le \|A\|_{\mathrm{F}} \|B\|_{\mathrm{F}},$$

and consistent/compatible with the Euclidean vector norm (2-norm), that is,

$$\|Ax\|_2 \le \|A\|_{\mathrm{F}} \|x\|_2.$$