NTNU

Problem Set 7

1 Solve the linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 4 & 20 & 8 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 3 \\ 20 \end{bmatrix},$$

by naïve Gauss-elimination and Gauss-elimination with partial pivoting.

2 This task demonstrates how a badly conditioned problem can be improved by simple means—in this case, at least—and should be carried out in MATLAB.

The net domestic production of crude oil in Norway from 1986 to 2010 measured in standard cubic metres ($\mathrm{S\,m^3}$) is provided in Table 1.

Table 1: Norwegian oil production in 1986–2010. Source: Statistics Norway.

Year	Oil production $(10^6\ \mathrm{S\,m^3})$
1986	48.771
1990	94.542
1994	146.282
1998	168.744
2002	173.649
2006	136.577
2010	104.354

a) Find the interpolation polynomial of degree 6 for the points in the table with help of the MATLAB-command `polyfit`. Notice that it complains about the polynomial being badly conditioned.

b) Next, repeat the exercise, but change the time axis by counting the years from 1986. Do you still get problems?

c) In order to find the coefficients of the polynomial p_n interpolating a collection of points $\{(t_i, y_i)\}_{i=0}^{n}$, MATLAB solves the linear system

$$a_n t_i^n + a_{n-1} t_i^{n-1} + \cdots + a_1 t_i + a_0 = y_i, \qquad i = 0, \ldots, n$$

with respect to a_i's. Set up the corresponding coefficient-matrices for the cases in a) and b) and check the condition numbers of both. What do you observe?

Hint: Use the command `vander`.

3 Given the iteration scheme

$$4x_{k+1} = -x_k - y_k + z_k + 2,$$

$$6y_{k+1} = 2x_k + y_k - z_k - 1,$$

$$-4z_{k+1} = -x_k + y_k - z_k + 4,$$

prove that $\boldsymbol{x}^{(k)} = (x_k, y_k, z_k)$ converges to a limit $\boldsymbol{x}$ for all starting values $\boldsymbol{x}^{(0)}$ as $k \to \infty$. What is the limit $\boldsymbol{x}$?

4 Solve the following two systems of equations by Gauss–Seidel iterations:

a) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}.$ b) $\begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & -5 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}.$

Use $(0.1, 0.1, 0.1)$ as the starting point. First do a few iterations by hand and then apply the attached MATLAB-program `gs.m`. Comment on the results. Do they comply with theory?

5 The system

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 \\ -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 6 \\ -2 \\ 6 \end{bmatrix}$$

is to be solved by an SOR method as follows.
Comment: the matrix can be built quickly in MATLAB with `toeplitz([4 −1 0 −1 0 0])` and setting two elements to 0.

a) Find an optimal ω and the corresponding $\rho(B_\omega)$. If you prefer, use the attached MATLAB-function `rhoSOR.m`, and plot $\rho(B_\omega)$ as a function of ω.

b) Do 10 iterations with the optimal ω; you can for example put $\boldsymbol{x}^{(0)} = \boldsymbol{0}$. For each iteration, print the error $\|\boldsymbol{x}^{(k)} - \boldsymbol{x}\|_2$.

Hint: rewrite the routine `gs.m` to do SOR iterations.

c) Repeat b) using other values of ω, *e.g.* 1.0 and 1.3. How does this affect the rate of convergence observed in b)? Is this as expected? Find a value of ω for which $\rho(B_\omega) = 1$, and perform iterations with values of ω around this value. How do the results comply with theory?